Robust $H_{\infty}$ Filtering for 2-D Discrete Fornasini-Marchesini Systems

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Abstract

The robust $H_{\infty}$ filtering problem for two-dimensional (2-D) systems described by uncertain Fornasini-Marchesini models is studied. Attention focuses on the design of $H_{\infty}$ filters such that the filter error system is asymptotically stable and preserves a guaranteed $H_{\infty}$ performance. By using the homogeneous polynomially parameter-dependent approach and adding slack matrix variables, the coupling between the Lyapunov matrix and the system matrices is eliminated. Then, a linear matrix inequality (LMI)-based approach is developed for designing the $H_{\infty}$ filter. An illustrative example shows the effectiveness of this approach.

Key words — Robustness, 2-D systems, $H_{\infty}$ filtering, Linear Matrix Inequalities (LMI), Uncertain systems, Fornasini-Marchesini model.

1 Introduction

In recent years, the control and filtering problems for 2-D systems have drawn considerable attention, as 2-D systems have important applications in the areas of multidimensional digital filtering, image data processing and transmission, thermal process modeling, etc. A number of important results have been obtained so far. To mention a few, the stability analysis and stabilization for 2-D systems has been investigated in [5, 7, 8, 13, 14, 15, 17], the robust $H_{\infty}$ filtering for 2-D systems in [3, 4, 6, 9, 12, 19], the $H_{\infty}$ filtering for 2-D systems with time delays in [10, 11, 20] and the $H_{\infty}$ filtering for Markovian jump parameter systems in [16]. These previous results on robust $H_{\infty}$ filtering problem for 2-D systems are

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mostly based on the method of quadratic stability conditions, and are hence inevitably conservative, since the same Lyapunov functions are used for the entire uncertainty domain. To overcome this, one possible way that has been well-recognized is to consider a parameter-dependent Lyapunov function whose aim is to reduce the over design in the quadratic framework. The basic idea is to decouple the product terms between the Lyapunov matrix and system matrices by introducing slack matrix variables to the well-established linear matrix inequality (LMI) performance conditions.

Thus, we study in this paper, the problem of robust $H_\infty$ filtering for 2-D discrete system in Fornasini-Marchesini model with parameter uncertainties, that belong to polytopes. The key in our approach is to utilize slack variables and the polynomially parameter-dependent idea. The reported results are based on homogenous polynomially parameter-dependent matrices of on arbitrary degree. It is proved that as the degree grows, increasing precision is obtained, providing less conservative filter designs. The proposed condition include results in the quadratic framework (that entail fixed matrices for the entire uncertainty domain), and the linearly parameter-dependent framework (that use linear convex combinations of matrices) as special cases.

The theoretical results are given as LMIs conditions, which can be solved by standard numerical software, as illustrated in the example at the end of the paper.

**Notations**: The notation used throughout the paper is standard. The superscript $T$ stands for matrix transposition. The notation $P > 0$ means that $P$ is real symmetric and positive definite. $I$ is the identity matrix with appropriate dimension. In symmetric block matrices or long matrix expressions, we use an asterisk (*) to represent a term that is induced by symmetry, and diag[...] stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

## 2 Problem Description

Consider the 2-D discrete systems described by the Fornasini-Marchesini model:

\[
\begin{align*}
x(i+1, j+1) &= A_{i}x(i, j+1) + A_{2}x(i+1, j) + B_{1}w(i, j+1) + B_{2}w(i+1, j) \\
y(i, j) &= C_{i}x(i, j) + D_{i}w(i, j) \\
z(i, j) &= L_{i}x(i, j)
\end{align*}
\]

(1)

where $x(i, j) \in R^n$ is the state vector, $y(i, j) \in R^m$ is the measured output vector, and $z(i, j) \in R^p$ is the signal to be estimated, $w(i, j) \in R^q$ is the disturbance input vector which belongs to $L_2\{[0, \infty), [0, \infty)\}$. The system
matrices are supposed to be unknown but belong to a given convex bounded
polyhedral domain, namely

\[ \Omega_\alpha = (A_{1\alpha}, A_{2\alpha}, B_{1\alpha}, B_{2\alpha}, C_\alpha, D_\alpha, L_\alpha) \in \mathcal{R} \]

\[ \mathcal{R} = \{ \Omega_\alpha | \Omega_\alpha = \sum_{i=1}^{s} \alpha_i \Omega_i; \alpha \in \Gamma \} \]  (2)

with \( \Omega_i = (A_{1i}, A_{2i}, B_{1i}, B_{2i}, C_i, D_i, L_i) \) denoting the vertices of the
polytope, and \( \Gamma \) the unit simplex:

\[ \Gamma = \{ (\alpha_1, \alpha_2, ..., \alpha_s) : \sum_{i=1}^{s} \alpha_i = 1, \alpha_i > 0 \} \]. (3)

The boundary condition of the state vector is supposed to be

\[ \lim_{n \to \infty} \sum_{k=1}^{n} (|x(0, k)|^2 + |x(k, 0)|^2) < \infty. \] (4)

Here, we are interested in estimating the signal \( z(i, j) \) by a robust filter of the
form

\[ \tilde{x}(i + 1, j + 1) = A_{f1} \tilde{x}(i, j + 1) + A_{f2} \tilde{x}(i + 1, j) + B_{f1} y(i, j + 1) + B_{f2} y(i + 1, j) \]

\[ \tilde{z}(i, j) = C_{f} \tilde{x}(i, j) \]  (5)

where \( \tilde{x}(i, j) \in \mathbb{R}^{n_f} \) is the state vector of the filter, and \( \tilde{z}(i, j) \in \mathbb{R}^p \) is the estimation of \( z(i, j) \).

If the augmented state vector is \( \xi(i, j) = [x^T(i, j) \; \tilde{x}^T(i, j)]^T \) and the estimation error is \( e(i, j) = z(i, j) - \tilde{z}(i, j) \), then the filtering error system can be
written as follows:

\[ \xi(i + 1, j + 1) = \tilde{A}_\alpha \xi(i, j) + \tilde{B}_\alpha \tilde{w}(i, j) \]

\[ \tilde{e}(i, j) = \tilde{C}_\alpha \xi(i, j) \]  (6)

where

\[ \tilde{A}_\alpha = \begin{bmatrix} A_{1\alpha} & A_{2\alpha} \\ B_{f1} C_\alpha & A_{f1} \end{bmatrix}, \quad \tilde{B}_\alpha = \begin{bmatrix} B_{1\alpha} & B_{2\alpha} \\ B_{f1} D_\alpha & B_{f2} D_\alpha \end{bmatrix}, \quad \tilde{C}_\alpha = \begin{bmatrix} L_\alpha - C_f \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ L_\alpha - C_f \end{bmatrix} \].

and

\[ \tilde{\xi}(i, j) = \begin{bmatrix} \xi(i, j + 1) \\ \xi(i + 1, j) \end{bmatrix}, \quad \tilde{w}(i, j) = \begin{bmatrix} w(i, j + 1) \\ w(i + 1, j) \end{bmatrix}, \quad \tilde{e}(i, j) = \begin{bmatrix} e(i, j + 1) \\ e(i + 1, j) \end{bmatrix} \]. (7)
The transfer function of the filtering error system is then
\[ \|T_{ew}(z_1, z_2, \alpha)\| = \bar{C}_\alpha [z_1 z_2 I_{2n} - z_2 \bar{A}_1 \alpha - z_1 \bar{A}_2 \alpha]^{-1} [z_2 \bar{B}_1 \alpha + z_1 \bar{B}_2 \alpha] \] (8)

Thus, the robust \( H_\infty \) filtering error problem can be given as follows.

**Problem description.** Given the system in (1) subject to parameter uncertain in (2), determine the filter of the form (5), such that the filter error system (6) is robustly asymptotically stable for all \( \alpha \in \Gamma \) and satisfies
\[ \|T_{ew}(z_1, z_2, \alpha)\|_\infty < \gamma \quad (\forall \alpha \in \Gamma) \] (9)

where \( \gamma \) is a given positive scalar and \( \|T_{ew}(z_1, z_2, \lambda)\|_\infty < \gamma \) is defined as the \( H_\infty \) performance of the filtering error system in (6).

Before deriving our main results, we give the following Lemmas.

**Lemma 2.1.** [1] Given the 2-D FM system in (1) and the filter in (5), for any fixed \( \alpha \in \Gamma \), the filtering error system in (6) is asymptotically stable and satisfies (9) if there exist matrices \( P_\alpha \in \mathbb{R}^{r \times r} > 0 \) and \( S_\alpha \in \mathbb{R}^{r \times r} > 0 \) satisfying
\[
\begin{bmatrix}
-R_\alpha & \bar{A}_\alpha^T P_\alpha & 0_{2r \times 2m} & \bar{C}_\alpha^T \\
* & -P_\alpha & P_\alpha \bar{B}_\alpha & 0_{r \times 2p} \\
* & * & -\gamma^2 I_{2m} & 0_{2m \times 2p} \\
* & * & * & -I_{2p}
\end{bmatrix} < 0
\] (10)

where \( R_\alpha = \text{diag}\{P_\alpha - S_\alpha, S_\alpha\} \) and \( r = n + n_f \)

**Lemma 2.2.** [18] Let \( \xi \in \mathbb{R}^n, Q \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{m \times n} \) with rank \( (B) = n \) and \( (B) \perp \) such that \( BB^\perp = 0 \). Then, the following conditions are equivalent:

1. \( \xi^T Q \xi < 0, \forall \xi \neq 0 : B \xi = 0 \)
2. \( B^T Q B^\perp < 0 \)
3. \( \exists \mu \in \mathbb{R} : Q - \mu B^T B < 0 \)
4. \( \exists \lambda \in \mathbb{R}^{n \times m} : Q + \lambda B + B^T \lambda^T < 0 \)

### 3 \( H_\infty \) Filtering Analysis

In this section, we assume that the filter matrices in (5) are known and we will study the condition under which the filter error system (6) is asymptotically stable with \( H_\infty \)-norm bounded \( \gamma \). Based on Lemma 2.1, we devote ourselves to the design of robust \( H_\infty \) filters. We first give the following proposition.
Proposition 3.1. Given the 2-D FM system in (1) and the filter (5), for any fixed $\alpha \in \Gamma$, the filtering error system in (6) is asymptotically stable and satisfies (9) if there exist matrices $P_\alpha \in \mathbb{R}^{r \times r} > 0$, $S_\alpha \in \mathbb{R}^{r \times r} > 0$, $K_\alpha \in \mathbb{R}^{2r \times r}$, $E_\alpha \in \mathbb{R}^{r \times r}$, $Q_\alpha \in \mathbb{R}^{2m \times r}$, $F_\alpha \in \mathbb{R}^{2p \times r}$, $G_\alpha \in \mathbb{R}^{2r \times 2p}$, $H_\alpha \in \mathbb{R}^{r \times 2p}$, $M_\alpha \in \mathbb{R}^{2m \times 2p}$, and $N_\alpha \in \mathbb{R}^{2p \times 2p}$, satisfying

$$
\begin{bmatrix}
\Gamma_1 & -K_\alpha + \bar{A}_\alpha E_\alpha^T + \bar{C}_\alpha H_\alpha^T & K_\alpha \bar{B}_\alpha + \bar{A}_\alpha Q_\alpha^T + \bar{C}_\alpha M_\alpha^T & -G_\alpha + \bar{A}_\alpha F_\alpha^T + \bar{C}_\alpha N_\alpha^T + \bar{C}_\alpha \\
* & P_\alpha - E_\alpha - E_\alpha^T & E_\alpha \bar{B}_\alpha - Q_\alpha^T & -H_\alpha - F_\alpha^T \\
* & * & Q_\alpha \bar{B}_\alpha + F_\alpha^T Q_\alpha^T - \gamma^2 I_{2m} & \bar{B}_\alpha^T F_\alpha^T - M_\alpha \\
* & * & \gamma^2 I_{2m} & -I_{2p} - N_\alpha - N_\alpha^T
\end{bmatrix} < 0 \quad (11)
$$

where $\Gamma_1 = K_\alpha \bar{A}_\alpha + \bar{A}_\alpha^T K_\alpha^T + G_\alpha C_\alpha + \bar{C}_\alpha^T G_\alpha^T - R_\alpha$.

Proof. The equivalence is obtained by considering

$$Q = \begin{bmatrix}
-R_\alpha & 0_{2r \times r} & 0_{2r \times 2m} & C^T \\
0_{r \times 2r} & P_\alpha & 0_{r \times 2p} & 0 \times 2p \\
0_{2m \times 2r} & P_\alpha & 0_{2m \times 2m} & -\gamma^2 I_{2m} \\
C & 0_{2m \times 2r} & 0_{2m \times 2m} & -I_{2p}
\end{bmatrix}, \quad B = \begin{bmatrix}
\bar{A}_\alpha & -I_r & \bar{B}_\alpha & 0_{r \times 2p} \\
\bar{C}_\alpha & 0_{2p \times r} & 0_{2p \times 2m} & -I_{2p}
\end{bmatrix},
$$

$$X = \begin{bmatrix}
X_\alpha & Y_\alpha
\end{bmatrix}, \text{ where } X_\alpha = \begin{bmatrix}
K_\alpha \\
E_\alpha \\
Q_\alpha \\
F_\alpha
\end{bmatrix}, \in \mathbb{R}^{(3r+2m+2p) \times r},$$

$$Y_\alpha = \begin{bmatrix}
G_\alpha \\
H_\alpha \\
M_\alpha \\
N_\alpha
\end{bmatrix}, \in \mathbb{R}^{(3r+2m+2p) \times (2p)},$$

in condition (4) of Lemma 2.2, with

$$B^\perp = \begin{bmatrix}
I_{2r} & 0_{2r \times 2m} \\
A_\alpha & B_\alpha \\
0_{2m \times 2r} & I_{2m} \\
C_\alpha & 0_{2p \times 2m}
\end{bmatrix}$$

Using condition (2) of Lemma 2.2, this gives condition (10), which completes the proof.

Remark 3.2. In Proposition 3.1, the slack variables $X_\alpha$ and $Y_\alpha$ are introduced. By setting $Y_\alpha = 0$, Proposition 3.1 coincides with the previous results for 1-D discrete systems [2]. Thus, Proposition 3.1 would generally render a
less conservative evaluation the upper bound of the $H_\infty$ norm in 2-D systems FM model case, which can be seen from the numerical example later in the paper.

4 $H_\infty$ Filter Design

In this section, a procedure will be established for designing a $H_\infty$ filter in (5), that is, to determine the filter matrices in (5) such that the filter error system (6) is asymptotically stable with $H_\infty$-norm bounded $\gamma$.

Based on Proposition 3.1, we select for variables $K_{\alpha}$, $E_{\alpha}$, $Q_{\alpha}$ and $F_{\alpha}$ the following structures [2, 18]:

$$I(i,j) = \begin{bmatrix} \lambda_i I & 0 \\ 0 & \lambda_j J \end{bmatrix}, \bar{K} = \begin{bmatrix} \hat{K} \\ \hat{K} \end{bmatrix}, \bar{F}_{\alpha} = \begin{bmatrix} F_{1\alpha} \\ F_{2\alpha} \end{bmatrix}, K_{11\alpha} = \begin{bmatrix} K_{1\alpha} \\ K_{2\alpha} \end{bmatrix}, K_{21\alpha} = \begin{bmatrix} K_{3\alpha} \\ K_{4\alpha} \end{bmatrix},$$

$$\bar{Q}_{\alpha} = \begin{bmatrix} Q_{1\alpha} \\ Q_{2\alpha} \end{bmatrix}, E_{\alpha} = \begin{bmatrix} E_{1\alpha} \\ E_{2\alpha} \end{bmatrix}, K_{\alpha} = \begin{bmatrix} K_{11\alpha} & I(2,3) \hat{K} \\ K_{21\alpha} & I(4,5) \hat{K} \end{bmatrix},$$

$$Q_{\alpha} = \begin{bmatrix} Q_{\alpha} \\ 0 \end{bmatrix}, F_{\alpha} = \begin{bmatrix} \hat{F}_{\alpha} \\ 0 \end{bmatrix}, N_{\alpha} = 0, G_{\alpha} = 0, H_{\alpha} = 0, M_{\alpha} = 0.$$

The matrices $P_{\alpha}$ and $S_{\alpha}$ are also partitioned in $n \times n$ blocks as follows

$$P_{\alpha} = \begin{bmatrix} P_{1\alpha} & P_{2\alpha} \\ P_{2\alpha}^T & P_{3\alpha} \end{bmatrix}, \quad S_{\alpha} = \begin{bmatrix} S_{1\alpha} & S_{2\alpha} \\ S_{2\alpha}^T & S_{3\alpha} \end{bmatrix},$$

and the following change of variables is adopted:

$$\begin{bmatrix} \hat{A}_f \\ \hat{B}_f \end{bmatrix} = \begin{bmatrix} \hat{K} & 0 \\ 0 & \hat{K} \end{bmatrix} \begin{bmatrix} A_{f1} & 0 \\ A_{f2} & 0 \end{bmatrix},$$

where $E_{1\alpha}$, $E_{2\alpha}$, $K_{11\alpha}$, $K_{21\alpha}$, $Q_{\alpha}$, and $F_{\alpha}$ are supposed to depend only on the parameter $\alpha$, while $\hat{K}$ is supposed to be fixed for the entire uncertainty domain and, without loss of generality, invertible. The scalar parameters $\lambda_i$, $i = 1...5$, will be searched through the entire uncertainty domain as part of optimization problems.

Theorem 4.1. If there exist symmetric parameter-dependent positive definite matrices $P_{\alpha}$, $S_{\alpha}$ as in (13) and parameter-dependent matrices $K_{\alpha}$, $E_{\alpha}$, $Q_{\alpha}$, and $F_{\alpha}$ as in (12), $\hat{A}_{f1}$, $\hat{A}_{f2}$, $\hat{B}_{f1}$, $\hat{B}_{f2}$, $\hat{C}_f$, $\gamma > 0$ and scalars $\lambda_i$,
\[ i = 1, \ldots, 5, \text{ such that} \]
\[
\begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} & \Psi_{16} & \Psi_{17} & \Psi_{18} & \Psi_{19} & A_{1a}^T F_{1a}^T \\
* & \Psi_{22} & \Psi_{23} & \Psi_{24} & \hat{A}_{f2}^T - K_{2a} & \Psi_{26} & \Psi_{27} & \Psi_{28} & -C_{f}^T & 0 \\
* & * & \Psi_{33} & \Psi_{34} & \Psi_{35} & \Psi_{36} & \Psi_{37} & \Psi_{38} & A_{2a}^T F_{1a}^T & \Psi_{310} \\
* & * & * & \Psi_{44} & \hat{A}_{f2}^T - K_{4a} & \Psi_{46} & \Psi_{47} & \Psi_{48} & 0 & -\hat{C}_{f}^T \\
* & * & * & * & \Psi_{55} & \Psi_{56} & \Psi_{57} & \Psi_{58} & -F_{1a}^T & -F_{2a}^T \\
* & * & * & * & * & \Psi_{66} & \Psi_{67} & \Psi_{68} & 0 & 0 \\
* & * & * & * & * & * & \Psi_{77} & \Psi_{78} & B_{1a}^T F_{1a}^T & B_{2a}^T F_{1a}^T \\
* & * & * & * & * & * & * & \Psi_{88} & B_{2a}^T F_{1a}^T & B_{2a}^T F_{2a}^T \\
* & * & * & * & * & * & * & * & -I & 0 \\
* & * & * & * & * & * & * & * & * & -I
\end{bmatrix} < 0 \tag{14}
\]

where

\[
\Psi_{11} = K_{1a} A_{1a} + A_{1a}^T K_{1a}^T + \lambda_2 (\kappa \hat{A}_{f1} C_a + C_{T} B_{T1}^T f_{T1}) + S_{1a} - P_{1a}
\]
\[
\Psi_{15} = A_{1a}^T E_{1a}^T + C_{T} B_{T1}^T f_{T1} - K_{1a}
\]
\[
\Psi_{17} = K_{1a} B_{1a} + \lambda_2 \hat{B}_{f1} D_a + A_{1a}^T Q_{1a}^T
\]
\[
\Psi_{19} = A_{1a}^T F_{1a}^T + L_{1a}^T
\]
\[
\Psi_{23} = K_{2a} A_{2a} + \lambda_1 \hat{B}_{f2} C_a + \lambda_1 \hat{A}_{f1}
\]
\[
\Psi_{26} = \lambda_1 \hat{A}_{f1} - \lambda_3 K
\]
\[
\Psi_{29} = \lambda_4 \hat{A}_{f2} + \lambda_3 \hat{B}_{f2}^T D_a + A_{2a}^T Q_{2a}
\]
\[
\Psi_{44} = \lambda_5 (\hat{A}_{f2} + \hat{A}_{f1}^T) - S_{3a}
\]
\[
\Psi_{47} = K_{4a} B_{2a} + \lambda_5 \hat{B}_{f2} D_a
\]
\[
\Psi_{55} = P_{1a} - E_{1a} + E_{1a}^T
\]
\[
\Psi_{57} = E_{1a} B_{1a} + \hat{B}_{f1} D_a - Q_{1a}^T
\]
\[
\Psi_{66} = P_{3a} - \lambda_1 (\hat{K} + \hat{K}^T)
\]
\[
\Psi_{68} = E_{2a} B_{2a} + \lambda_1 \hat{B}_{f2} D_a
\]
\[
\Psi_{78} = Q_{1a} B_{2a} + \hat{B}_{f2}^T Q_{2a}
\]
\[
\Psi_{33} = K_{3a} A_{2a} + A_{2a}^T K_{3a}^T + \lambda_4 (\hat{B}_{f2} C_a + C_{T} B_{T1}^T f_{T1}) + S_{1a}
\]
\[
\Psi_{22} = \lambda_3 (\hat{A}_{f1} + \hat{A}_{f1}^T) + S_{3a} - P_{3a}
\]
\[
\Psi_{24} = \lambda_3 \hat{A}_{f2} + \lambda_5 \hat{A}_{f1}
\]
\[
\Psi_{27} = K_{2a} B_{1a} + \lambda_3 \hat{B}_{f1} D_a
\]
\[
\Psi_{30} = A_{2a}^T F_{2a}^T + L_{2a}^T
\]
\[
\Psi_{46} = \lambda_1 \hat{A}_{f2} - \lambda_5 K
\]
\[
\Psi_{48} = K_{4a} B_{2a} + \lambda_5 \hat{B}_{f2} D_a
\]
\[
\Psi_{56} = P_{2a} - \hat{K} - E_{2a}^T
\]
\[
\Psi_{58} = E_{1a} B_{2a} + \hat{B}_{f2} D_a - Q_{2a}^T
\]
\[
\Psi_{67} = E_{2a} B_{1a} + \lambda_1 \hat{B}_{f1} D_a
\]
\[
\Psi_{77} = Q_{1a} B_{1a} + B_{1a}^T Q_{1a}^T - \gamma^2 I
\]
\[
\Psi_{13} = K_{1a} A_{2a} + \lambda_2 \hat{B}_{f2} C_a + A_{2a}^T K_{3a} + \lambda_4 C_{T} B_{T1}^T f_{T1}
\]
\[
\Psi_{88} = Q_{2a} B_{2a} + B_{2a}^T Q_{2a} - \gamma^2 I
\]
\[
\Psi_{14} = \lambda_2 \hat{A}_{f2} + A_{1a}^T K_{4a} + \lambda_5 C_{T} B_{T1}^T f_{T1}
\]
holds for all $\alpha \in \Gamma$, then
\[
\begin{bmatrix}
A_{f1} & B_{f1} \\
A_{f2} & B_{f2} \\
C_f & 0
\end{bmatrix}
= \begin{bmatrix}
\hat{K}^{-1} & 0 & I \\
0 & \hat{K}^{-1} & 0 \\
\hat{A}_{f1} & \hat{B}_{f1} & 0 \\
\hat{A}_{f2} & \hat{B}_{f2} & 0 \\
0 & 0 & \hat{C}_f
\end{bmatrix}
\] (17)
are the matrices of the robust stable filter that ensure a guaranteed cost $H_\infty$ given by $\gamma$.

**Remark 4.2.** When the scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\lambda_5$ of Theorem 4.1 are fixed to be constants, then (14) is an LMI (it is linear in the variables). To select values for these scalars, optimization can be used (for example fminsearch in MATLAB) to improve some performance measures, such as the disturbance attenuation level $\gamma$.

**Remark 4.3.** In order to solve the parameter-dependent LMI conditions of Theorem 4.1, the technique proposed in [2] to handle parameter-dependent LMIs with parameters in the unit simplex can be applied. For this, the decision variables $P_{1a}, P_{2a}, P_{3a}, S_{1a}, S_{2a}, S_{3a}, E_{1a}, E_{2a}, K_{1a}, K_{2a}, K_{3a}, K_{4a}, Q_{1a}, Q_{2a}, F_{1a}$ and $F_{2a}$) are treated as homogenous polynomials of arbitrary degree $g$, so the corresponding LMI conditions, that although sufficient are increasingly precise when increasing $g$, are expressed just in terms of the vertices of the polytope.

## 5 Numerical Example

Consider a 2-D static field model described by the differential equation:
\[
\eta_{i,j+1} = \alpha_1 \eta_{i,j} + \alpha_2 \eta_{i+1,j} - \alpha_1 \alpha_2 \eta_{i,j} + \omega_{1(i,j)}
\] (18)
where $\eta_{i,j}$ is the state of coordinates $(i, j)$, and $\alpha_1, \alpha_2$ are the vertical and horizontal correlative coefficients respectively, satisfying $\alpha_1^2 < 1$ and $\alpha_2^2 < 1$. Defining the augmented state vector $x_{i,j} = [\eta_{i,j+1}^T - \alpha_2 \eta_{i+1,j}^T]^T$, and supposing that the measured equation and the signal to be estimated are
\[
y_{i,j} = \alpha_1 \eta_{i,j+1} + (1 - \alpha_1 \alpha_2) \eta_{i+1,j} + \omega_2 \\
z_{i,j} = \eta_{i,j}
\] (19)

it is not difficult to transform the above equations into a 2-D FM model in the form of (1), with the corresponding system matrices given by
\[
A_{1a} = \begin{bmatrix}
\alpha_1 & 0 \\
0 & 0
\end{bmatrix},
A_{2a} = \begin{bmatrix}
0 & 0 \\
1 & \alpha_2
\end{bmatrix},
B_{1a} = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix},
B_{2a} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
C_a = \begin{bmatrix}
\alpha_1 & 1
\end{bmatrix},
D_a = \begin{bmatrix}
0 & 1
\end{bmatrix},
H_a = \begin{bmatrix}
0 & 1
\end{bmatrix}.
\]
where we assume that $0.15 \leq \alpha_1 \leq 0.45$, and $0.35 \leq \alpha_2 \leq 0.85$: therefore the above system can be represented by a four-vertex polytopic system.

By solving LMI (14), the results obtained are given in table 1. In this Example, we show that less conservative designs are achieved as the degree of the polynomial grows, when applying the HPPD approach.

For degree $g = 0$, the $H_\infty$ disturbance attenuation level is $\gamma = 3.8691$ (with $\lambda_1 = 1.0074$, $\lambda_2 = 0.0000$, $\lambda_3 = 0.0001$, $\lambda_4 = 0.0000$, $\lambda_5 = 0.0000$, obtained by optimization following Remark 4.2), and the filter matrices are

\[
A_{f1} = \begin{bmatrix} 0.2772 & -0.0631 \\ 0.0647 & -0.0147 \end{bmatrix}, A_{f2} = \begin{bmatrix} 0.0992 & -0.0082 \\ 0.8269 & -0.0683 \end{bmatrix}, B_{f1} = \begin{bmatrix} -0.0626 \\ -0.0146 \end{bmatrix}
\]

\[
B_{f2} = \begin{bmatrix} -0.0752 \\ -0.7663 \end{bmatrix}, C_f = \begin{bmatrix} 0.0001 & -1.0075 \end{bmatrix}.
\]

For degree $g = 1$ (linearly parameter dependent approach), we get $\gamma = 2.4883$ (with $\lambda_1 = 1.8950$, $\lambda_2 = 0.0497$, $\lambda_3 = 0.0588$, $\lambda_4 = 0.1118$, $\lambda_5 = -0.1922$) and the filter matrices are

\[
A_{f1} = \begin{bmatrix} 0.5711 & -0.1615 \\ 0.0403 & -0.0134 \end{bmatrix}, A_{f2} = \begin{bmatrix} -0.0889 & 0.0110 \\ 0.2529 & 0.2846 \end{bmatrix}, B_{f1} = \begin{bmatrix} -0.1155 \\ -0.0098 \end{bmatrix}
\]

\[
B_{f2} = \begin{bmatrix} 0.0268 \\ -0.3841 \end{bmatrix}, C_f = \begin{bmatrix} -0.0348 & -1.4971 \end{bmatrix}.
\]

For degree $g = 2$, $\gamma = 2.4880$ (with $\lambda_1 = 2.2051$, $\lambda_2 = 0.0428$, $\lambda_3 = 0.0552$, $\lambda_4 = 0.1077$, $\lambda_5 = -0.2072$), and the filter matrices are

\[
A_{f1} = \begin{bmatrix} 0.5973 & -0.1540 \\ 0.0412 & -0.0122 \end{bmatrix}, A_{f2} = \begin{bmatrix} -0.0868 & 0.0085 \\ 0.2542 & 0.2867 \end{bmatrix}, B_{f1} = \begin{bmatrix} -0.0947 \\ -0.0077 \end{bmatrix}
\]

\[
B_{f2} = \begin{bmatrix} 0.0219 \\ -0.3237 \end{bmatrix}, C_f = \begin{bmatrix} -0.0586 & -1.7711 \end{bmatrix}.
\]

<table>
<thead>
<tr>
<th>Degree g</th>
<th>Theorem 4.1</th>
<th>Theorem 3[1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.8691</td>
<td>3.8709</td>
</tr>
<tr>
<td>1</td>
<td>2.4883</td>
<td>2.5450</td>
</tr>
<tr>
<td>2</td>
<td>2.4880</td>
<td>2.5028</td>
</tr>
</tbody>
</table>

Table 1: Comparisons of disturbance levels obtained for the Example with [1] and the proposed approach.
The maximum singular values curve of the filtering error transfer function

![Singular Value Curve](image)

Figure 1: Singular value curve of the filtering error system in the Example, with the filter (5), for $(\alpha_1, \alpha_2) = (0.15, 0.35)$, from the results using a polynomial of degree $g = 2$.

with the filter in Eq.(5) in Example for different vertices are given in Table 2. The results obtained for this example using the proposed approach clearly show the improvement with respect to previous results in the literature.

| $\alpha_1$ | 0.15 | 0.15 | 0.45 | 0.45 |
| $\alpha_2$ | 0.35 | 0.85 | 0.35 | 0.85 |
| $\gamma_{\min}$ | 1.5092 | 1.5267 | 1.7528 | 1.7373 |

Table 2: $H_\infty$ norms at the vertices of Theorem 4.1 corresponding to degree $g = 2$

**6 Conclusion**

In this article, we have investigated the $H_\infty$ filtering problem for 2-D discrete-time systems described by an uncertain Fornasini-Marchesini model. The Lyapunov function approach was used, so that by adding slack matrix variables, a new LMI-based condition for $H_\infty$ performance analysis has been proposed. A numerical example is used to illustrate the effectiveness of the proposed method.
Figure 2: Singular value curve of the filtering error system in the Example, with the filter (5), for \((\alpha_1, \alpha_2) = (0.15, 0.85)\), from the results using a polynomial of degree \(g = 2\)

Figure 3: Singular value curve of the filtering error system in the Example, with the filter (5), when \((\alpha_1, \alpha_2) = (0.45, 0.35)\), from the results using a polynomial of degree \(g = 2\)
Figure 4: Singular value curve of the filtering error system in the Example, with the filter in Eq.(5), when $(\alpha_1, \alpha_2) = (0.45, 0.85)$, from the results using a polynomial of degree $g = 2$

References


