Stability and stabilization of 2D continuous time varying delay systems

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Abstract. This paper deals with the problem of delay-dependent stability and stabilization of 2D continuous time varying delay systems described by the Roesser model. Some sufficient conditions ensuring asymptotic stability and stabilization are established in forms of linear matrix inequality (LMI) technique via Lyapunov techniques with additional free weighting matrices. A numerical example is introduced to show the efficiency of the proposed criteria for a 2D linear time-varying delay system.

Keywords: 2D Roesser model, time-varying delay systems, Lyapunov-Krasovskii functional, LMI.

1 Introduction

Delayed multidimensional systems have been recently introduced but in the majority of the existing studies only the discrete case have been analyzed (see e.g. [5, 10, 11, 15, 16]) except for a few recent papers [1, 7, 9] where a Lyapunov approach is applied to continuous Roesser models. These papers consider a constant time delays. Recently, the delay-dependent stability problem for two dimensional systems with time-varying delays has been addressed [4, 13], in the discrete case. However, to the author’s knowledge, in the continuous case, this problem has not been fully investigated except a recent paper we have published in [6] where a delay dependent stability criterion is derived for 2D continuous time varying delay systems. It is inspired from ([14, 17]) where some delay-dependent stability criteria, for one dimensional continuous systems, are devised by taking into account the relationship between the terms in the Leibniz-Newton formula by means of a set of free weighting matrices leading to linear matrix inequalities (LMI) conditions.

This paper addresses the problem of stability and stabilization for 2D continuous time varying delay systems. The paper is organized as follows: in section 2, we introduce the mathematical background needed to address the problem. In section 3, we introduce our main results: first, a sufficient condition is derived to check the asymptotic stability of the system using Lyapunov techniques. This delay dependent condition is different from the one presented in [6] and will be shown to be less conservative. In the derivative of the Lyapunov functional, the term
\[ \dot{x}(t_1, t_2) \] is retained but the relationship among the term in the system equation is expressed by some free weighting matrices. In consequence, the Lyapunov matrices in the Lyapunov functional are not involved in any product terms with the system matrices. This idea developed in [8] provides some extra freedom in the selection of the weighting matrices, which have the potential to yield less conservative results. Second, we give a delay dependent criterion to design a state feedback controller for 2D continuous time varying delay systems which stabilizes the system. These conditions are expressed in terms of LMIs (linear matrix inequalities, see [3]). Finally section 4 presents an illustrative example to show the effectiveness of the proposed criteria.

**Notations:**
Throughout the paper we will use the following notations: a matrix added to its symmetric will be called sym \{ A \} = A^T + A and (\ast) in a symmetric matrix denotes the corresponding symmetric element. Also, \( 0_{n \times m} \) is the \( n \times m \) zero matrix, and \( I_n \) is the \( n \times n \) identity matrix. Some formula will be used in the paper, in particular the Leibniz-Newton formula which is given by

\[
x^h(t_1, t_2) - x^h(t_1 - \tau_1(t_1), t_2) - \int_{t_1 - \tau_1(t_1)}^{t_1} \dot{x}^h(s, t_2) ds = 0 \tag{1}
\]

**2 Problem formulation**

The class of 2-D systems with delays under consideration is represented by an extension of the Roesser model (see [12] and [2]) of the form:

\[
\begin{bmatrix}
\frac{\partial x^h(t_1, t_2)}{\partial t_1} \\
\frac{\partial x^v(t_1, t_2)}{\partial t_2}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
x^h(t_1, t_2) \\
x^v(t_1, t_2)
\end{bmatrix} +
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u(t_1, t_2) \tag{2}
\]

where \( x^h(t_1, t_2) \) is the horizontal state in \( \mathbb{R}^{n_h} \), \( x^v(t_1, t_2) \) is the vertical state in \( \mathbb{R}^{n_v} \), \( u(t_1, t_2) \) is the control vector in \( \mathbb{R}^m \), \( \tau_1 \) and \( \tau_2 \) are the delays in horizontal and vertical directions respectively and \( A_{ij}, A_{ijd} \) and \( B_i, (i,j = 1,2) \), are real constant matrices of appropriate dimensions. The initial conditions are given by

\[
x^h(\theta, t_2) = f(\theta, t_2), \forall t_2 \quad \text{and} \quad -h_1 < \theta < 0
\]

\[
x^v(t_1, \theta) = g(t_1, \theta), \forall t_1 \quad \text{and} \quad -h_2 < \theta < 0
\]

The time-delays \( \tau_1(t_1) \) and \( \tau_2(t_2) \) are time-varying continuous functions that satisfy

\[
0 < \tau_1(t_1) \leq h_1, \tau_1(t_1) \leq d_1
\]

\[
0 < \tau_2(t_2) \leq h_2, \tau_2(t_2) \leq d_2
\]
where $f$ and $g$ are continuous functions. For such a system we denote

$$x(t_1, t_2) \equiv \begin{bmatrix} x^h(t_1, t_2) \\ x^0(t_1, t_2) \end{bmatrix}, \quad \dot{x}(t_1, t_2) \equiv \begin{bmatrix} \frac{\partial x^h(t_1, t_2)}{\partial t_1} \\ \frac{\partial x^0(t_1, t_2)}{\partial t_2} \end{bmatrix}$$

and

$$x(t_1 - \tau_1(t_1), t_2 - \tau_2(t_2)) \equiv \begin{bmatrix} x^h(t_1 - \tau_1(t_1), t_2) \\ x^0(t_1, t_2 - \tau_2(t_2)) \end{bmatrix}$$

which allows us to write (2) in the usual form

$$\dot{x}(t_1, t_2) = Ax(t_1, t_2) + A_d x(t_1 - \tau_1(t_1), t_2 - \tau_2(t_2)) + Bu(t_1, t_2)$$

(3)

Consider the state feedback control:

$$u(t_1, t_2) = K x(t_1, t_2)$$

(4)

where the matrix

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

is the state feedback gain to be determined.

### 3 Main results

#### 3.1 Asymptotic stability

In this section, we investigate stability condition for time-varying delay system (2), with $u(t_1, t_2) = 0$.

**Theorem 1** Given matrices $H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} > 0$, $U = H^{-1}$ and $W = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} < I$, the system (2) is asymptotically stable if there exist symmetric positive-definite matrices

$$P = P^T = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} > 0, \quad Q = Q^T = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} > 0$$

and any appropriately dimensioned matrices $Y_0, Y_1, Y_2, A_1, A_2$ and $A_3$ such that the following LMI is verified:

$$\Phi = \begin{bmatrix} Q - \text{sym} \{A_1 A_1^T\} + \text{sym} \{Y_0\} & A^T A_2^T - A_1 A_d + Y_{10} \\ \ast & -(I_n - W)Q - \text{sym} \{A_2 A_d\} - \text{sym} \{Y_1\} \end{bmatrix} < 0$$

(5)

with

$$Y_0 = \begin{bmatrix} S_0 & 0 \\ 0 & T_0 \end{bmatrix}; \quad Y_1 = \begin{bmatrix} S_1 & 0 \\ 0 & T_1 \end{bmatrix}; \quad Y_2 = \begin{bmatrix} S_2 & 0 \\ 0 & T_2 \end{bmatrix}; \quad Y_{10} = \begin{bmatrix} S_1 - S_0^T & 0 \\ 0 & T_1 - T_0^T \end{bmatrix}$$

$$A_1 = \begin{bmatrix} N_1 & 0 \\ 0 & M_1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} N_2 & 0 \\ 0 & M_2 \end{bmatrix}; \quad A_3 = \begin{bmatrix} N_3 & 0 \\ 0 & M_3 \end{bmatrix}$$
Proof. Proof of theorem 1 is given in the appendix 6.1.

3.2 Stabilization

The objective of this section is the design of a stabilizing state-feedback controller for system (2). Using the state-feedback control (4), (2) can be rewritten as:

\[ \dot{x}(t_1, t_2) = A_c x(t_1, t_2) + A_d x(t_1 - \tau_1(t_1), t_2 - \tau_2(t_2)) + Bu(t_1, t_2) \] (6)

where:

\[ A_c = \begin{bmatrix} (A_{11} + B_1 K_1) & (A_{12} + B_1 K_2) \\ (A_{21} + B_2 K_1) & (A_{22} + B_2 K_2) \end{bmatrix} \]

\[ A_d = \begin{bmatrix} A_{11d} & A_{12d} \\ A_{21d} & A_{22d} \end{bmatrix} \]

The problem is then to compute a static feedback control given by (4) such that the closed-loop 2D system (6) is asymptotically stable.

Theorem 2 Let \( H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} > 0 \), \( U = H^{-1} \) and \( W = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} < I \) be given matrices, then the system (2) is stabilizable with the control law (4) if there exist symmetric positive-definite matrices \( X = X^T = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} > 0 \), \( \bar{P} = \bar{P}^T = \begin{bmatrix} \bar{P}_0 & 0 \\ 0 & \bar{P}_2 \end{bmatrix} > 0 \), \( \bar{Q} = \bar{Q}^T = \begin{bmatrix} \bar{Q}_1 & 0 \\ 0 & \bar{Q}_2 \end{bmatrix} > 0 \) and \( \bar{R} = \bar{R}^T = \begin{bmatrix} \bar{R}_1 & 0 \\ 0 & \bar{R}_2 \end{bmatrix} > 0 \) and any appropriately dimensioned matrices \( \bar{T}_0, \bar{T}_1, \bar{T}_2 \) and \( Y > 0 \) such that the following LMI is verified:

\[
\begin{bmatrix}
Q + \text{sym} \{\bar{T}_0\} - \text{sym} \{AX\} - \text{sym} \{BY\} & -A_d X - X A^T - Y^T B^T + \bar{T}_0^T & -(I_n - W)Q - \text{sym} \{\bar{T}_1\} - \text{sym} \{A_d X\} \\
* & * & * \\
\bar{T}_2^T + X + \bar{P} - X A^T - Y^T B^T & H U \bar{T}_0 & < 0 \\
X - X A_d^T - \bar{T}_2^T & H U \bar{T}_1 & \\
H \bar{R} + \text{sym} \{X\} & H U \bar{T}_2 & \\
* & -H \bar{U} \bar{R} &
\end{bmatrix}
\]

with

\[
\begin{bmatrix}
\bar{T}_0 \\
\bar{T}_1 \\
\bar{T}_2 \\
\bar{T}_{10}
\end{bmatrix} =
\begin{bmatrix}
\bar{S}_0 & 0 & 0 & 0 \\
0 & \bar{T}_0 & 0 & 0 \\
0 & 0 & \bar{T}_2 & 0 \\
0 & 0 & 0 & \bar{T}_{10}
\end{bmatrix}
\]

and \( X = A^{-1} \), \( Y = K X, Q = \Lambda \bar{Q} A, R = \Lambda \bar{R} A, P = \Lambda \bar{P} A, \bar{T}_i = \Lambda \bar{T}_i A, \) \( i = 0, 1, 2 \).

The gains \( K_1 \) and \( K_2 \) of the control law (4) are given by

\[
K_1 = Y_1 X_1^{-1}, K_2 = Y_2 X_2^{-1}
\] (8)

Proof. Proof of Theorem 2 is given in the appendix 6.2.
4 Example

In order to show the applicability of our results, consider a 2D continuous system represented by (2) with:

\[
A_{11} = \begin{bmatrix}
-1.8887 & -1.4069 \\
-0.1447 & -2.1601
\end{bmatrix},
A_{22} = \begin{bmatrix}
2.2169 & -1.0753 \\
6.0811 & 0.9372
\end{bmatrix},
\]
\[
A_{12} = \begin{bmatrix}
15.8162 & -6.7649 \\
4.2121 & 5.0797
\end{bmatrix},
A_{21} = \begin{bmatrix}
-0.7902 & 0.011 \\
-0.4672 & -1.7982
\end{bmatrix},
\]

The delay matrices are given by:

\[
A_{11d} = \begin{bmatrix}
-0.1 & 0 \\
-0.1 & -0.1
\end{bmatrix},
A_{22d} = \begin{bmatrix}
-0.9 & 0 \\
1 & -1.1
\end{bmatrix},
\]
\[
A_{12d} = \begin{bmatrix}
0.4 & 0.4 \\
-0.08 & 0.04
\end{bmatrix},
A_{21d} = \begin{bmatrix}
-0.24 & 0 \\
0 & 0.04
\end{bmatrix},
\]
\[
B_1 = \begin{bmatrix}
1 & 0.3 \\
0 & 0.5
\end{bmatrix},
B_2 = \begin{bmatrix}
0.1 & 0 \\
0.2 & 0.3
\end{bmatrix}
\]

The parameters \(h_1\), \(h_2\), \(d_1\) and \(d_2\) are modified in an iterative process until the LMI (7) was found feasible. The obtained feedback gains

\[
K_1 = \begin{bmatrix}
-0.1185 & 0.0132 \\
1.8912 & 0.5704
\end{bmatrix},
K_2 = \begin{bmatrix}
-40.0278 & 10.7805 \\
-46.1860 & -29.8926
\end{bmatrix}
\]

are then injected to construct the closed loop system which is again checked by condition (5) of Theorem 1. The maximum bounds of delays obtained are

\[
h_{1max} = 8.37, \quad h_2 = 3.33, \quad \text{and} \quad d_1 = d_2 = 0.8
\]

The condition obtained in [6] applied to the present closed loop system yields the delay bounds \(h_{1max} = 0.3273\) and \(h_2 = 0.3026\) which illustrate that the delay-dependent condition given in this paper is less conservative than the existing result proposed in [6].

Remark 1. From a numerical point of view, it is worth noting that matrices \(A_{12}\), \(A_{21}\), \(A_{12d}\) and \(A_{21d}\) could yield badly conditioned LMI’s. In addition, the fact that all the matrices are block diagonal increases the possibility that the LMI will be badly conditioned resulting in non feasible LMI.

5 Conclusion

To conclude, let us highlight the general contribution of this paper. We first developed a sufficient condition of asymptotic stability for 2D continuous time varying delay systems. Using Lyapunov approach and the Leibniz-Newton formula, we proposed the synthesis of a state feedback controller. The interesting fact in these conditions is that they are delay dependent and expressed in terms of LMIs, so they are tractable from a computational point of view. Finally, a numerical example is provided to illustrate the results.
6 Appendix

6.1 Proof of Theorem 1

Lemma 1 ([14]). For any semi-positive definite matrix $X^h = \begin{bmatrix} X^h_{11} & 0 & X^h_{12} & 0 & X^h_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & X^h_{22} & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & X^h_{33} & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}$, the following holds

$$0, \text{ the following holds}$$

\begin{align*}
\int_{t_1}^{t_1} \xi^T(t_1, t_2) X^h \xi(t_1, t_2) - \int_{t_1 - \tau_1(t_1)}^{t_1} \xi^T(t_1, t_2) X^h \xi(t_1, t_2) \geq 0 
\end{align*}

(9)

where

\begin{align*}
\xi(t_1, t_2) &= [x^h(t_1, t_2) \ x^v(t_1, t_2) \ x^h(t_1 - \tau_1(t_1), t_2) \ x^v(t_1, t_2 - \tau_2(t_2)) \\
\dot{x}^h(t_1, t_2) &= \dot{x}^v(t_1, t_2)]^T
\end{align*}

Let us define

$$V(x(t_1, t_2)) = V_1(t_1, t_2) + V_2(t_1, t_2)$$

(10)
as a possible Lyapunov Krasovskii functional candidate for the system (2) with:

\begin{align*}
V_1(t_1, t_2) &= x^h(t_1, t_2) P_1 x^h(t_1, t_2) + \int_{t_1 - \tau_1(t_1)}^{t_1} x^h(t_1 - \tau_1(t_1), t_2) Q_1 x^h(t_1 - \tau_1(t_1), t_2) d\theta \\
&\quad + \int_{-\tau_1}^{0} \int_{t_1 + \theta}^{t_1} \dot{x}^h(s, t_2) R_1 \dot{x}^h(s, t_2) d\theta d\theta \\
V_2(t_1, t_2) &= x^v(t_1, t_2) P_2 x^v(t_1, t_2) + \int_{t_2 - \tau_2(t_2)}^{t_2} x^v(t_1, \theta) Q_2 x^v(t_1, \theta) d\theta \\
&\quad + \int_{-\tau_2}^{0} \int_{t_2 + \theta}^{t_2} \dot{x}^v(t_1, k) R_2 \dot{x}^v(t_1, k) d\theta d\theta
\end{align*}

The derivative of function $V(x(t_1, t_2))$ along the vector

$$\xi(t_1, t_2) = \begin{bmatrix} \frac{\partial x^h(t_1, t_2)}{\partial t_1} \\ \frac{\partial x^h(t_1, t_2)}{\partial t_2} \end{bmatrix}$$

is given by:

\begin{align*}
\nabla \xi V(x(t_1, t_2)) &= (\nabla V)^T \xi(t_1, t_2) \\
&= \frac{\partial V_1(t_1, t_2)}{\partial x^h(t_1, t_2)} \frac{\partial x^h(t_1, t_2)}{\partial t_1} + \frac{\partial V_2(t_1, t_2)}{\partial x^h(t_1, t_2)} \frac{\partial x^h(t_1, t_2)}{\partial t_2} \\
&\quad + \frac{\partial V_1(t_1, t_2)}{\partial x^v(t_1, t_2)} \frac{\partial x^v(t_1, t_2)}{\partial t_1} + \frac{\partial V_2(t_1, t_2)}{\partial x^v(t_1, t_2)} \frac{\partial x^v(t_1, t_2)}{\partial t_2}
\end{align*}
where $\nabla V$ is the gradient of the function $V$. Let

\[
\xi(t_1, t_2) = \begin{bmatrix} x^T(t_1, t_2) & x^T(t_1 - \tau_1(t), t_2) & x^T(t_1, t_2 - \tau_2(t)) \\
\dot{x}^T(t_1, t_2) & \dot{x}^T(t_1 - \tau_1(t), t_2) & \dot{x}^T(t_1, t_2 - \tau_2(t)) \end{bmatrix}^T,
\]

\[
\zeta(t_1, t_2, s, k) = \begin{bmatrix} x^T(t_1, t_2) & x^T(t_1 - \tau_1(t), t_2) & x^T(t_1, t_2 - \tau_2(t)) \\
\dot{x}^T(t_1, t_2) & \dot{x}^T(t_1 - \tau_1(t), t_2) & \dot{x}^T(t_1, t_2 - \tau_2(t)) \\
\dot{x}^T(s, t_2) & \dot{x}^T(s, t_2) & \dot{x}^T(s, t_2) \end{bmatrix}^T
\]

and

\[
e_i = \begin{bmatrix} 0_{n \times (i-1)n} I_n 0_{n \times (8-i)n} \end{bmatrix}^T, i = 1, 2, ..., 8
\]

Using the Leibniz-Newton formula (1), we can write

\[
x^h(t_1 - \tau_1(t), t_2) = x^h(t_1, t_2) - \int_{t_1-\tau_1(t)}^{t_2} \dot{x}^h(s, t_2) ds
\]

Then, for any appropriately dimensioned matrices $S_0$, $S_1$ and $S_2$, we have

\[
2 \begin{bmatrix} x^T(t_1, t_2)S_0 + x^T(t_1 - \tau_1(t), t_2)S_1 + \dot{x}^T(t_1, t_2)S_2 \end{bmatrix} \times
\]

\[
\left\{ x^h(t_1, t_2) - x^h(t_1 - \tau_1(t), t_2) - \int_{t_1-\tau_1(t)}^{t_2} \dot{x}^h(s, t_2) ds \right\} = 0
\]

Similarly, for any matrices $N_1$, $N_2$ and $N_3$ of appropriate dimensions, we have

\[
2 \begin{bmatrix} x^T(t_1, t_2)N_1 + x^T(t_1 - \tau_1(t), t_2)N_2 + \dot{x}^T(t_1, t_2)N_3 \end{bmatrix} \times
\]

\[
\left\{ \dot{x}^h(t_1, t_2) - A_{11}x^h(t_1, t_2) - A_{12}\dot{x}^h(t_1, t_2) - A_{13}x^h(t_1, t_2 - \tau_2(t)) \right\} = 0.
\]

The free weighting matrices $S_i$ $(i = 0, 1, 2)$ in (12) are used to express the relationship between $x^h(t_1, t_2)$, $x^h(t_1 - \tau_1(t), t_2)$ and $\int_{t_1-\tau_1(t)}^{t_2} \dot{x}^h(s, t_2) ds$, using the Leibniz-Newton formula.

The free weighting matrices $N_i$ $(i = 1, 2, 3)$ in (13) are used to take into account the model of the system, that is, the relation between $\dot{x}^h(t_1, t_2)$, $x^h(t_1, t_2)$ and $x^h(t_1 - \tau_1(t), t_2)$. The key idea behind is to consider $\dot{x}(t_1, t_2)$ as a variable in the first derivative of the Lyapunov-Krasovskii functional.

Computing the derivative of $V_i(t_1, t_2)$ along the trajectories of (2) gives:

\[
\frac{\partial V_i(t_1, t_2)}{\partial x^h(t_1, t_2)} \frac{\partial x^h(t_1, t_2)}{\partial t_1} = 2x^h(t_1, t_2)P_i \dot{x}^h(t_1, t_2) + x^h(t_1, t_2)Q_i x^h(t_1, t_2)
\]

\[
- (1 - \tau_1(t_1))x^h(\tau_1(t_1), t_2)Q_i x^h(t_1 - \tau_1(t_1), t_2) + h_1 x^h(t_1, t_2)R_i \dot{x}^h(t_1, t_2)
\]

\[
- \int_{t_1-h_1}^{t_1} x^h(s, t_2) R_i \dot{x}^h(s, t_2) ds
\]
Then, for any matrices \( S_0, S_1, S_2, N_1, N_2 \) and \( N_3 \) using (12) and (13), we can bound the derivative as follows:

\[
\frac{\partial V_1(t_1, t_2)}{\partial x^h(t_1, t_2)} \frac{\partial x^h(t_1, t_2)}{\partial t_1} \leq \\
2x^{hT}(t_1, t_2)P_1\dot{x}^h(t_1, t_2) + x^{hT}(t_1, t_2)Q_1x^h(t_1, t_2) \\
- (1 - d_1)x^{hT}(t_1 - \tau_1(t_1), t_2)Q_1x^h(t_1 - \tau_1(t_1), t_2) + R_1\dot{x}^h(t_1, t_2) \\
- \int_{t_1 - \tau_1(t_1)}^{t_1} x^{hT}(s, t_2)R_1\dot{x}^h(s, t_2)ds \\
+ 2\left[ x^{hT}(t_1, t_2)S_0 + x^{hT}(t_1 - \tau_1(t_1), t_2)S_1 + x^{hT}(t_1, t_2)S_2 \right] \times \\
\left\{ x^h(t_1, t_2) - x^h(t_1 - \tau_1(t_1), t_2) - \int_{t_1 - \tau_1(t_1)}^{t_1} \dot{x}^h(s, t_2)ds \right\} \\
+ 2\left[ x^{hT}(t_1, t_2)N_1 + x^{hT}(t_1 - \tau_1(t_1), t_2)N_2 + x^{hT}(t_1, t_2)N_3 \right] \times \\
\left\{ x^h(t_1, t_2) - A_{111}x^h(t_1, t_2) - A_{122}\dot{x}^h(t_1 - \tau_1(t_1), t_2) - A_{124}\dot{x}^h(t_1, t_2 - \tau_2(t_2)) \right\}
\]

Further, using the newly defined vector \( \xi(t_1, t_2) \), the expression above can be rewritten as

\[
\frac{\partial V_1(t_1, t_2)}{\partial x^h(t_1, t_2)} \frac{\partial x^h(t_1, t_2)}{\partial t_1} \leq \\
\xi^T(t_1, t_2) \left\{ 2e_2^T P_1 e_5 + e_1^T Q_1 e_1 - (1 - d_1)e_3^T Q_1 e_3 + h_1 e_5^T R_1 e_5 \right\} \xi(t_1, t_2) \\
- \int_{t_1 - \tau_1(t_1)}^{t_1} \xi(t_1, t_2, s, k)^T e_7^T R_1 e_7 \xi(t_1, t_2, s, k)ds \\
+ 2\xi^T(t_1, t_2) \left\{ e_1^T S_0 e_1 - e_2^T S_0 e_3 + e_3^T S_1 e_1 - e_4^T S_1 e_3 + e_5^T S_2 e_1 - e_5^T e_2 e_3 \right\} \xi(t_1, t_2) \\
- 2\int_{t_1 - \tau_1(t_1)}^{t_1} \xi(t_1, t_2, s, k)^T \left\{ e_1^T S_0 e_7 + e_2^T S_1 e_7 + e_5^T S_2 e_7 \right\} \xi(t_1, t_2, s, k)ds \\
+ 2\xi^T(t_1, t_2) \left\{ e_1^T N_1 e_5 - e_1^T N_1 A_{111} e_1 - e_4^T N_1 A_{112} e_2 - e_5^T N_1 A_{114} e_3 - e_5^T N_1 A_{124} e_4 \\
e_1^T N_2 e_5 - e_1^T N_2 A_{111} e_1 - e_4^T N_2 A_{112} e_2 - e_5^T N_2 A_{114} e_3 - e_5^T N_2 A_{124} e_4 \\
e_1^T N_3 e_5 - e_1^T N_3 A_{111} e_1 - e_4^T N_3 A_{112} e_2 - e_5^T N_3 A_{114} e_3 - e_5^T N_3 A_{124} e_4 \right\} \xi(t_1, t_2).
We get similar expression for the second direction, which combined with the inequality obtained above yield the following condition for both directions

\[
\nabla_c V(x(t_1, t_2)) \leq \xi^T(t_1, t_2) \left\{ 2e_2^TP_1e_5 + e_1^TQ_1e_1 - (1-d_1)e_1^TQ_1e_3 + h_1e_5^TR_1e_5 \right\} \xi(t_1, t_2)
\] 
\[- \int_{t_1-\tau_1(t_1)}^{t_1} \zeta(t_1, t_2, s, k)^T e_1^T R_1e_1 \zeta(t_1, t_2, s, k)ds
\] 
\[+ 2\xi^T(t_1, t_2) \left\{ e_1^TS_0e_1 - e_1^TS_0e_3 + e_1^TS_1e_1 - e_1^TS_1e_3 + e_2^TS_2e_1 - e_2^TS_2e_3 \right\} \xi(t_1, t_2)
\] 
\[- 2\int_{t_1-\tau_1(t_1)}^{t_1} \zeta(t_1, t_2, s, k)^T \left\{ e_1^TS_0e_7 + e_1^TS_1e_7 + e_1^TS_2e_7 \right\} \zeta(t_1, t_2, s, k)ds
\] 
\[+ 2\xi^T(t_1, t_2) \left\{ e_1^TN_1e_5 - e_1^TN_1A_{11}e_1 - e_1^TN_1A_{12}e_2 - e_1^TN_1A_{11}A_{12}e_3 - e_1^TN_1A_{11}A_{12}e_4
\]
\[e_1^TN_2e_5 - e_1^TN_2A_{11}e_1 - e_1^TN_2A_{12}e_2 - e_1^TN_2A_{11}A_{12}e_3 - e_1^TN_2A_{11}A_{12}e_4
\]
\[e_1^TN_3e_5 - e_1^TN_3A_{11}e_1 - e_1^TN_3A_{12}e_2 - e_1^TN_3A_{11}A_{12}e_3 - e_1^TN_3A_{11}A_{12}e_4 \right\} \xi(t_1, t_2)
\] 
\[+ \xi^T(t_1, t_2) \left\{ e_2^TP_2e_6 + e_2^TQ_2e_2 - (1-d_2)e_3^TQ_1e_3 + h_2e_6^TR_2e_6 \right\} \xi(t_1, t_2)
\] 
\[- \int_{t_2-\tau_2(t_2)}^{t_2} \zeta(t_1, t_2, s, k)^T e_3^TR_2e_8 \zeta(t_1, t_2, s, k)ds
\] 
\[+ 2\xi^T(t_1, t_2) \left\{ e_2^T T_0e_2 - e_2^T T_0e_4 + e_4^T T_1e_2 - e_4^T T_1e_4 + e_6^T T_2e_2 - e_6^T T_2e_4 \right\} \xi(t_1, t_2)
\] 
\[- 2\int_{t_2-\tau_2(t_2)}^{t_2} \zeta(t_1, t_2, s, k)^T \left\{ e_2^T T_0e_8 + e_4^T T_1e_8 + e_6^T T_2e_8 \right\} \zeta(t_1, t_2, s, k)ds
\] 
\[2\xi^T(t_1, t_2) \left\{ e_4^T M_1e_6 - e_2^T M_1A_{11}e_1 - e_2^T M_1A_{21}e_3 - e_2^T M_1A_{22}e_4 - e_2^T M_1A_{22}e_4
\]
\[e_4^T M_2e_6 - e_4^T M_2A_{11}e_1 - e_4^T M_2A_{21}e_3 - e_4^T M_2A_{22}e_4 - e_4^T M_2A_{22}e_4
\]
\[e_6^T M_3e_6 - e_6^T M_3A_{21}e_1 - e_6^T M_3A_{22}e_2 - e_6^T M_3A_{22}e_3 - e_6^T M_3A_{22}e_4 \right\} \xi(t_1, t_2).
\]
Then, applying lemma 1, respectively for $X^h$ and $X_v$, we get

$$
\begin{align*}
\dot{H}x(t_1, t_2)X\xi(t_1, t_2) - \int_{t_1-\tau_1(t_1)}^{t_1} \xi^T(t_1, t_2)X^h\xi(t_1, t_2)ds - \int_{t_1, t_2-\tau_2(t_2)}^{t_2} \xi^T(t_1, t_2)X^v\xi(t_1, t_2)dk & \\
= \dot{H}x(t_1, t_2)X\xi(t_1, t_2) - \\
\int_{t_1-\tau_1(t_1)}^{t_1} \int_{t_1, t_2-\tau_2(t_2)}^{t_2} \xi^T(t_1, t_2) \left\{ \text{diag} \left( \begin{array}{ccc} 1 & 0 & 0 \\ \frac{1}{\tau_2(t_2)} & 0 & 0 \\ 0 & \frac{1}{\tau_2(t_2)} & 0 \end{array} \right) \right\} X^h & \\
+ \text{diag} \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{\tau_1(t_1)} & 0 & 0 \\ 0 & \frac{1}{\tau_1(t_1)} & 0 \end{array} \right\} X^v \xi(t_1, t_2)ds \ & \\
\leq \dot{H}x(t_1, t_2)X\xi(t_1, t_2) - \\
\int_{t_1-\tau_1(t_1)}^{t_1} \int_{t_1, t_2-\tau_2(t_2)}^{t_2} \xi^T(t_1, t_2) \left\{ \text{diag} \left( \begin{array}{ccc} \frac{1}{h_2} & 0 & 0 \\ 0 & \frac{1}{h_2} & 0 \\ 0 & 0 & \frac{1}{h_2} \end{array} \right) \right\} X^h & \\
+ \text{diag} \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{h_1} & 0 & 0 \\ 0 & \frac{1}{h_1} & 0 \end{array} \right\} X^v \xi(t_1, t_2)ds 
\end{align*}
$$

\begin{equation}
\dot{H}x^T(t_1, t_2)X\xi(t_1, t_2) - \int_{t_1-\tau_1(t_1)}^{t_1} \int_{t_2-\tau_2(t_2)}^{t_2} \xi^T(t_1, t_2)\bar{U}X\xi(t_1, t_2)ds \tag{14}
\end{equation}

with

$$
\bar{U} = \text{diag} \{ U \quad U \quad U \}
$$

\begin{equation}
\nabla_x V(x(t_1, t_2)) \leq \xi^T(t_1, t_2)\Xi\xi(t_1, t_2) - \int_{t_1-\tau_1(t_1)}^{t_1} \int_{t_2-\tau_2(t_2)}^{t_2} \xi^T(t_1, t_2, s, k)\Psi\xi(t_1, t_2, s, k)dsdk
\end{equation}

with

$$
\Xi = \begin{bmatrix}
Q - \text{sym} \{ A_1A_1 \} + \text{sym} \{ Y_0 \} + HX_{11} & A^T_1A^T_2 - A_1A_2 + Y_{10} + HX_{12} & \ast \\
-\ast & -\ast & -\ast \\
A_2 - A^T_2A^T_3 + \gamma_3^T + HX_{13} & H\bar{R} + \text{sym} \{ A_3 \} + HX_{23} & < 0
\end{bmatrix}
$$

and

$$
\Psi = \begin{bmatrix}
UX_{11} & UX_{12} & UX_{13} & UY_0 \\
\ast & UX_{22} & UX_{23} & UY_1 \\
\ast & \ast & UX_{33} & UY_2 \\
\ast & \ast & \ast & UR
\end{bmatrix}
$$

If $\Xi < 0$ and $\Psi > 0$, then $\nabla_x V(x(t_1, t_2)) \prec -\epsilon \lVert x(t_, t_2) \rVert^2$ for a sufficiently small $\epsilon$, which ensures the asymptotic stability of system (2).

Specifically, if we select $R > 0$ then $X$ can be chosen to be

$$
X = \begin{bmatrix}
Y_0 \\
Y_1 \\
Y_2
\end{bmatrix}
$$

$$
R^{-1} \begin{bmatrix}
Y_0^T & Y_1^T & Y_2^T
\end{bmatrix} \geq 0.
$$

This ensures that $\Psi > 0$. In this case, $\Xi < 0$ is equivalent to $\Phi < 0$ according to the Schur complement.
6.2 Proof of Theorem 2

Using the same Lyapunov functional as mentioned in section 6.1, for the closed loop system, we get the condition

\[
\Phi_c = \begin{bmatrix}
Q - \text{sym} \{ A_1 A_c \} + \text{sym} \{ Y_0 \} & A^T_1 A_c - A_1 A_d + Y_{10} \\
* & - (I_n - W) Q - \text{sym} \{ A_2 A_d \} - \text{sym} \{ Y_1 \}
\end{bmatrix} < 0,
\]

that is, according to Theorem 1, is sufficient to ensure that the closed loop system is stable.

Consider the case where \( A_1 = A_2 = A_3 = A \), then

\[
\Phi_c = \begin{bmatrix}
Q - \text{sym} \{ AA_c \} + \text{sym} \{ Y_0 \} & A^T_1 A - A A_d + Y_{10} \\
* & - (I_n - W) Q - \text{sym} \{ AA_d \} - \text{sym} \{ Y_1 \}
\end{bmatrix} < 0,
\]

Note that this last condition is bilinear with respect to the variables \( A \) and \( K \) and therefore it may be considered as a BMI problem. To obtain LMI (7), it is necessary to pre- and post-multiply inequality (15) by \( \text{diag} \{ A^{-1}, A^{-1}, A^{-1}, A^{-1} \} \).

Références


