International Journal of Sciences and Techniques of Automatic control & computer engineering
IJ-STA, Volume 2, N° 1, July 2008, pp. 484–499.

Adaptive observer design for a class of nonlinear systems with coupled structures

T. Mâatoug[1,2], M. Farza[2], M. M'Saad[2], Y. Koubaa[1], M. Kamoun[1]
[1]ENIS, Département de Génie électrique,
BP W, 3038 Sfax, Tunisia
[2]GREYC, UMR 6072 CNRS, Université de Caen, ENSICAEN
6 Boulevard Maréchal Juin, 14050 Caen Cedex, France
maatougtarak@yahoo.fr, mfarza@greyc.ensicaen.fr

Abstract: In this paper, we propose a global exponential adaptive observer for a class of uniformly observable nonlinear systems in order to jointly estimate missing states and unknown constant parameters. This class consists of cascade subsystems where every sub-system is associated with a subset of outputs. Moreover, a full triangular structure is not assumed since the dynamics of some particular states of each subsystem may depend on the whole state vector. Of fundamental importance, the global exponential convergence of the proposed observers was shown to be guaranteed under the well known persistent excitation condition. The gain of this observer involves a design function that has to satisfy some mild conditions which are given. Different expressions of such a function are proposed. Of particular interest, it is shown that adaptive high gain like observers and adaptive sliding mode like observers can be derived by considering particular expressions of the design function.
Keywords: Nonlinear system, High gain observer, Sliding mode, Adaptive observer, Persistent excitation.

1 Introduction

During the last two decades, the adaptive observer design problem for MIMO nonlinear systems has received much attention in the literature and motivated a lot of work, for adaptive control, or recently fault detection and isolation in dynamic systems. Various results are available for linear systems can be found in [13, 18]while more recent results are reported in [22, 23]. Since the eighties, many results on nonlinear systems have became available. For example, adaptive observers have been proposed for a class of nonlinear systems which can be linearized with a change of coordinates up to output injection in [1, 16, 17, 15]. Their applicability is limited by the restrictive linearization condition. More recently,some more general results on nonlinear systems have been reported in [19, 4, 2]. These methods

This paper was recommended for publication in revised form by the editor Staff.
Edition: CPU of Tunis, Tunisia, ISSN: 1737-7749

do not require the considered nonlinear systems to be linearizable, instead, they assume the existence of some Lyapounov function satisfying particular conditions. They are not constructive methods in the sense in that there is no systematic way for the design of the required Lyapounov function, and it is not known how to systematically check their applicability to a given system. In a relatively recent work [21], the authors proposed an adaptive observer for a class of single output uniformly observable nonlinear systems admitting some high gain observer, but further depending on unknown parameters. This consists in an extension of the approach initially proposed in [22] for MIMO linear time-varying systems. The main advantage of this approach lies in both design and implementation simplicities.
In this paper, one proposes to extend this approach to a large class of uniformly observable nonlinear MIMO systems . The class of systems in which the subsystems for each output has triangular dependence on the states of that subsystem itself except its last states dynamic which can depend on the whole states of systems is to be considered. To this end, one shall consider the following class of MIMO nonlinear systems.

$$\begin{cases} \dot{x} = Ax + g(u,x) + \Psi(u,x)\rho \\ y = Cx = x^1 \end{cases} \tag{1}$$

Then, one shall combine the approach adopted in [11] with those proposed in [23] and [22] in order to design an adaptive nonlinear observer. The main characteristics of the proposed observer lie in its simplicity and its capability to give rise to different observers among which adaptive high gain like observers and adaptive sliding mode like observers. Indeed, the gain of the state estimation as well as that of the parameter adaptation involve a design function that has to satisfy some mild conditions which are given. Different expressions of the design function are proposed and it is shown that adaptive high gain like observers [3, 10, 11, 7] and adaptive sliding mode like observers [20, 5, 6, 9] can be derived by considering particular expressions of the design function. Of particular interest, the tuning of the observer is achieved through the choice of a single parameter.
This paper is organized as follows. The next section presents the class of nonlinear MIMO systems which has a triangular form for each block, the observer design will be detailed and the observer equations are given. Different expressions of the design function are given to emphasize the versatility of the proposed adaptive observer in section 3.

2 Adaptive observer design

Consider a class of nonlinear MIMO systems which are equivalent by diffeomorphism to systems of the form:

$$\begin{cases} \dot{x} = Ax + g(u,x) + \Psi(u,x)\rho \\ y = Cx \end{cases} \tag{2}$$

where the state $x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix} \in \mathbb{R}^n$ with $x^k = \begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_{\lambda_k}^k \end{pmatrix} \in \mathbb{R}^{n_k}, x_i^k \in \mathbb{R}^{p_k}$ $k = 1, \ldots, q$

and $\sum_{k=1}^{p} n_k = n$; the output $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix} \in \mathbb{R}^p$ with $y_k \in \mathbb{R}^{p_k}$, $k = 1, \ldots, q$ and

$\sum_{k=1}^{q} p_k = p$; $A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_q \end{bmatrix}, A_k = \begin{bmatrix} 0 & I_{p_k} & & 0 \\ \vdots & & \ddots & \\ 0 & \ldots & 0 & I_{p_k} \\ 0 & \ldots & 0 & 0 \end{bmatrix}$; $C = \begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_q \end{bmatrix}$,

$C_k = \begin{bmatrix} I_{p_k} & 0 & \ldots & 0 \end{bmatrix}$, the nonlinear field $g(u,x) = \begin{pmatrix} g^1(u,x) \\ g^2(u,x) \\ \vdots \\ g^q(u,x) \end{pmatrix} \in \mathbb{R}^n$,

$g^k(u,x) = \begin{pmatrix} g_1^k(u,x) \\ g_2^k(u,x) \\ \vdots \\ g_{\lambda_k}^k(u,x) \end{pmatrix} \in \mathbb{R}^{n_k}$ where for $k = 1, \ldots, q$, the element g_i^k denotes i^{th} element of k^{th} nonlinear function $g^k(u,x)$ and has the structural dependance on the states:

• for $1 \leq i \leq \lambda_k - 1$

$$g_i^k(u,x) = g_i^k(u; x^1, x^2, \ldots, x^{k-1}; x_1^k, x_2^k, \ldots, x_i^k; x_1^{k+1}, x_1^{k+2}, \ldots, x_1^q) \qquad (3)$$

• for $i = \lambda_k$:

$$g_{\lambda_k}^k(u,x) = g_{\lambda_k}^k(u; x^1, x^2, \ldots, x^q) \qquad (4)$$

$$\rho = \begin{pmatrix} \rho^1 \\ \rho^2 \\ \vdots \\ \rho^q \end{pmatrix} \in R^m, \ with \ \rho^k = \begin{pmatrix} \rho_1^k \\ \rho_2^k \\ \vdots \\ \rho_m^k \end{pmatrix} \in R^{m_k}, \ k = 1, \ldots, q \ \ and \sum_{k=1}^{q} m_k = m.$$

ρ is the vector of unknown constant parameters.
$\Psi(u,x)$ is $n \times m$ matrix such that:

$$\Psi^T(u,x)=\begin{pmatrix}\Psi^{1^T}(u,x)\\ \Psi^{2^T}(u,x)\\ \vdots\\ \Psi^{q^T}(u,x)\end{pmatrix},\ \Psi^{k^T}(u,x)=\begin{pmatrix}\Psi_1^{k^T}(u,x)\\ \Psi_2^{k^T}(u,x)\\ \vdots\\ \Psi_s^{k^T}(u,x)\end{pmatrix},\Psi_s^k(u,x)=\begin{pmatrix}\Psi_s^{1k}(u,x)\\ \Psi_s^{2k}(u,x)\\ \vdots\\ \Psi_s^{\lambda_k k}(u,x)\end{pmatrix}.$$

$s = 1,\ldots, m_k$ and $\Psi_s^k(u,x)$ denotes s^{th} block of the matrix $\Psi^k(u,x)$ with:

$$\Psi_s^{ik}=(u;x^1,x^2,\ldots,x^{k-1};x_1^k,x_2^k,\ldots,x_i^k;x_1^{k+1},x_1^{k+2},\ldots,x_1^q),\ for\ 1\le i\le \lambda_k-1.$$

and it has the same structure like $g_i^k(u,x)$.
Our objective consists in designing adaptive observers to simultaneously estimate the state and the unknown parameters. Such a design requires some assumptions which will be stated in due courses. At this step, one assumes the following:
(A1) The functions $g(u,x)$ and $\Psi(u,x)$ are globally Lipschitz with respect to x uniformly in u.
(A2) The matrix $\Psi(u(t),x(t))$ is uniformly bounded.

2.1 Observer design

Before giving our candidate observer, one introduces the following notations:
1) let Δ_θ be the block diagonal matrix defined by:

$$\Delta_k(\theta)=\begin{bmatrix} I_{p_k} & & & \\ & \frac{1}{\theta^{\delta_k}}I_{p_k} & & \\ & & \ddots & \\ & & & \frac{1}{\theta^{\delta_k(\lambda_k-1)}}I_{p_k}\end{bmatrix} \tag{5}$$

where $\theta\ge 1$ is a parameter design and the δ_k's are defined as follows:

$$\begin{cases}\delta_k=\prod_{i=k+1}^{q}(\lambda_i-1)\delta\quad for\ \ 1\le k\le q-1;\\ \delta_q=\delta;\ \ \delta>0\quad is\ a\ real\ number.\end{cases} \tag{6}$$

2) let Λ_θ be the block diagonal matrix defined by:

$$\Lambda_k(\theta)=\begin{bmatrix}\frac{1}{\theta^{\sigma_1^k}}I_{p_k} & & \\ & \ddots & \\ & & \frac{1}{\theta^{\sigma_{\lambda_k}^k}}I_{p_k}\end{bmatrix} \tag{7}$$

where σ_i^k is given by:

$$\sigma_i^k=\sigma_1^k+(i-1)\delta_k\ for\ i\ =\ 1,\ldots,\lambda_k;k=1,\ldots,q \tag{8}$$

and $\sigma_1^k, k = 1, \ldots, q$ are integers which are chosen as follows [8]:

$$\sigma_1^k = (\lambda_1 - \frac{1}{2})\delta_1 - (\lambda_k - \frac{1}{2})\delta_k + (1 - \frac{1}{2^{k-1}}) \tag{9}$$

3) Let $S_{\theta^{\delta_k}}$ be the unique solution of the algebraic Lyapunov equation :

$$\theta^{\delta_k} S_{\theta^{\delta_k}} + A_k^T S_{\theta^{\delta_k}} + S_{\theta^{\delta_k}} A_k = C_k^T C_k \tag{10}$$

where

$$S_{\theta^{\delta_k}} = \frac{1}{\theta^{\delta_k}} \Delta_k(\theta) S_k \Delta_k(\theta) \tag{11}$$

where A_k and C_k are given in system (2) . It can be shown that the explicit solution of (10) is symmetric positive definite for every $\theta > 0$ and that :

$$S_k(i,j) = (-1)^{(i+j)} C_{i+j-2}^{j-1} I_{p_k} \;\; for \;\; 1 \le i, \; j \le n_k \;\; where \; C_n^p = \frac{n!}{(n-p)!p!} \tag{12}$$

In particular, $S_k^{-1} C_k^T = \begin{pmatrix} C_{n_k}^1 I_{p_k}, & C_{n_k}^2 I_{p_k}, & \ldots, & C_{n_k}^{n_k} I_{p_k} \end{pmatrix}^T$.

4) Let Ω_θ be a the following matrix:

$$\Omega_k(\theta) = \begin{bmatrix} \frac{1}{\theta^{\varepsilon_1^k}} & & \\ & \ddots & \\ & & \frac{1}{\theta^{\varepsilon_{m_k}^k}} \end{bmatrix} \tag{13}$$

where ε_j^k, are positive integers which are chosen such that each term of the matrix $\Delta_k(\theta)\Psi^k(u,x)\Omega_k^{-1}(\theta)$ is a polynomial in $\frac{1}{\theta}$ [14].

5) For any $\xi^k \in \mathbb{R}^{p_k q}$, Let $\Upsilon_\xi^k(t)$ be a $p_k q \times m_k$ matrix satisfying the following Ordinary Differential Equation (ODE):

$$\dot{\Upsilon}_\xi^k = \theta^{\delta_k} \left(\; (A_k - S_{1k}^{-1} C_k^T C_k) \Upsilon_\xi^k + \Delta_k(\theta) \Psi^k(u,\xi) \Omega_k^{-1}(\theta) \; \right) \tag{14}$$

6) Let $P_k(t)$ be the $m_k \times m_k$ symmetric matrix governed by the following differential equation:

$$\dot{P}_k(t) = -\theta^{\delta_k} \left(\; P_k(t) \Upsilon_\xi^{k^T}(t) C_k^T C_k \Upsilon_\xi^k(t) P_k(t) - P_k(t) \; \right) \tag{15}$$

where $P_k(t_0) \in R^{m_k} \times R^{m_k}$ is chosen symmetric positive definite and the matrix $\Upsilon_\xi^k(t)$ governed by (14).

7)$\forall \xi^k = \begin{pmatrix} \xi_1^k \\ \xi_2^k \\ \vdots \\ \xi_{\lambda_k}^k \end{pmatrix} \in \mathbb{R}^{n_k}$ $k = 1,\ldots,q$, set $\bar{\xi^k} = \Lambda_k(\theta)\xi^k$ and let

$K(\xi^k) \stackrel{\Delta}{=} K(\xi^{1k}) = \begin{pmatrix} K^1(\xi^{1k}) \\ \vdots \\ K^q(\xi^{1k}) \end{pmatrix} \in \mathbb{R}^{n_k}$, $k = 1,\ldots,q$ be a vector of smooth functions satisfying the following property:
Let D_k be any compact subset of $\mathbb{R}^{n_k}$, then

$$\forall \xi^k \in D_k : \bar{\xi^k}^T C_k^T C_k K(\xi^k) \quad \geq \quad \frac{1}{2}\bar{\xi^k}^T C_k^T C_k \xi^k \tag{16}$$

The observer synthesis needs the following additional assumptions :
(A3) For any $\xi \in \mathbb{R}^{n_k}$, the matrix $C_k \Upsilon_\xi^k(t)$ is persistently exciting.

Since the matrix $(A_k - S_{1k}^{-1} C_k^T C_k)$ is Hurwitz and each term of the matrix $\Delta_k(\theta)\Psi^k(u,x)\Omega_k^{-1}(\theta)$ is polynomial in $1/\theta$, one can conclude that for $\theta \geq 1$ the matrix $\Upsilon_\xi^k(t)$ is bounded and corresponding bounds do not depend on θ. Furthermore, one can show that under assumption (A6), the matrix $P_k(t)$ governed by (15) is symmetric positive definite and that it is bounded from above and from below and corresponding bounds do not depend on θ.

A candidate adaptive observer for system (2) is:

$$\dot{\hat{x}}(t) = A\hat{x} + g(u,\hat{x}) + \Psi(u,\hat{x})\hat{\rho}(t) - \Theta\Delta^{-1}(\theta)\left(S^{-1} + \Upsilon_{\hat{x}}(t)P(t)\Upsilon_{\hat{x}}^T(t)\right)C^T C\, K(\tilde{x}) \tag{17}$$

$$\dot{\hat{\rho}}(t) = \ -\Omega^{-1}(\theta)P(t)\Upsilon_{\hat{x}}^T(t)\Theta^2 C^T C\, K(\tilde{x}) \tag{18}$$

and

$$\begin{aligned} \dot{\hat{x}}^k(t) &= A_k\hat{x}^k + g^k(u,\hat{x}) + \Psi^k(u,\hat{x})\hat{\rho}^k(t) \\ &- \theta^{\delta_k}\Delta_k^{-1}(\theta)\left(S_k^{-1} + \Upsilon_{\hat{x}}^k(t)P_k(t)\Upsilon_{\hat{x}}^{k^T}(t)\right)C_k^T C_k K(\tilde{x^k}) \end{aligned} \tag{19}$$

$$\dot{\hat{\rho}}^k(t) = \ -\theta^{2\delta_k}\Omega_k^{-1}(\theta)P_k(t)\Upsilon_{\hat{x}}^{k^T}(t)C_k^T C_k K(\tilde{x}^k) \tag{20}$$

where S_Θ, $\Upsilon_{\hat{x}}(t)$, $p(t)$, $\Omega(\theta)$ and $\Delta(\theta)$ are respectively defined as follows:

$$S = \begin{pmatrix} S_1 & & \\ & \ddots & \\ & & S_q \end{pmatrix},\ \Upsilon_{\hat{x}}(t) = \begin{pmatrix} \Upsilon_{\hat{x}}^1(t) & & \\ & \ddots & \\ & & \Upsilon_{\hat{x}}^q(t) \end{pmatrix},$$

$$P(t) = \begin{pmatrix} P_1(t) & & \\ & \ddots & \\ & & P_q(t) \end{pmatrix},\ \Omega(\theta) = \begin{pmatrix} \Omega_1(\theta) & & \\ & \ddots & \\ & & \Omega_q(\theta) \end{pmatrix},$$

$$\Delta(\theta)=\begin{pmatrix}\Delta_1(\theta) & & \\ & \ddots & \\ & & \Delta_q(\theta)\end{pmatrix} \; and$$

$$\Theta = diag\left(\ \theta^{\delta_1}I_1,\quad \theta^{\delta_2}I_2,\quad \ldots,\quad \theta^{\delta_q}I_q\ \right)$$

where the states

$$\hat{x}=\begin{pmatrix}\hat{x}^1\\ \hat{x}^2\\ \vdots\\ \hat{x}^q\end{pmatrix}\in \mathbb{R}^{\,n},\;\; with\;\; \hat{x}^k=\begin{pmatrix}\hat{x}^k_1\\ \hat{x}^k_2\\ \vdots\\ \hat{x}^k_{\lambda_k}\end{pmatrix}\in \mathbb{R}^{\,n_k},$$

$k=1,\ldots,q$ and $\sum_{k=1}^{q} n_k = n$; $\tilde{x}=\hat{x}-x$ where x is the unknown trajectory of system (2); $\hat{\rho}=\begin{pmatrix}\hat{\rho}^1\\ \hat{\rho}^2\\ \vdots\\ \hat{\rho}^q\end{pmatrix}\in \mathbb{R}^m$; $K(\tilde{x})$ is a rectangular matrix governed by equation (16); u and y are respectively the input and the output of system (2).
Indeed, one states the following:

Theorem 2.1 *Assume that system (2) satisfies Assumptions (A1) to (A3). Then, system (17) is a global exponential adaptive observer for system (2).*

The proof of this theorem is detailed in section.

2.2 Convergence analysis

Set the estimation error $\tilde{x}(t)=\hat{x}(t)-x(t)$ and $\tilde{\rho}(t)=\hat{\rho}(t)-\rho$ then there dynamic equation are given by the following (for the raison of ambiguity, one omits the time t for each variable).

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x}+g(u,\hat{x})-g(u,x)+\Psi(u,\hat{x})\hat{\rho}+\Psi(u,x)\rho\\ &- \Theta\Delta^{-1}(\theta)\left(S^{-1}+\Upsilon_{\hat{x}}\;P\;\Upsilon_{\hat{x}}^T\right)C^TCK(\tilde{x})\\ &= A\tilde{x}+g(u,\hat{x})-g(u,x)+\Psi(u,x)\tilde{\rho}+\left(\Psi(u,\hat{x})-\Psi(u,x)\right)\rho\\ &- \Theta\Delta^{-1}(\theta)\left(S^{-1}+\Upsilon_{\hat{x}}\;P\;\Upsilon_{\hat{x}}^T\right)C^TCK(\tilde{x})\end{aligned} \tag{21}$$

$$\dot{\tilde{\rho}}(t)= -\Theta^2\Omega^{-1}(\theta)P(t)\Upsilon_{\hat{x}}^T(t)C^TCK(\tilde{x}) \tag{22}$$

where u is an admissible control such that $\|u\|_\infty \leq M$, $M > 0$ is a given constant.

The considered class of nonlinear system (2)has the block triangular structure. therefore, one has in particular fore k'th subsystem the following equation:

$$\begin{aligned}
\dot{\tilde{x}}^k &= A_k\tilde{x}^k + g^k(u,\hat{x}) - g^k(u,x) + \Psi^k(u,x)\tilde{\rho}^k + \left(\Psi^k(u,\hat{x}) - \Psi^k(u,x)\right)\rho^k \\
&- \theta^{\delta_k}\Delta_k^{-1}(\theta)\left(S_k^{-1} + \Upsilon_{\hat{x}}^k\ P_k\ \Upsilon_{\hat{x}}^{k^T}\right)C_k^T C_k K(\tilde{x}^k) \qquad (23)
\end{aligned}$$

$$\dot{\tilde{\rho}}^k(t) = -\theta^{2\delta_k}\Omega_k^{-1}(\theta)P_k(t)\Upsilon_{\hat{x}}^{k^T}(t)C_k^T C_k K(\tilde{x}^k) \qquad (24)$$

one can easily check the following identities:

- $\Lambda_k(\theta)A_k\Delta_k^{-1}(\theta) = \theta^{\delta_k}A_k$
- $\Lambda_k(\theta)\Delta_k^{-1}(\theta) = \theta^{-\sigma_1^k}I_k$ (I_k is the $n_k \times n_k$ identity matrix)
- $C_k\Lambda_k^{-1}(\theta) = \theta^{\sigma_1^k}C_k$
- $C_k\Delta_k^{-1}(\theta) = C_k$

set $\bar{x}^k = \Lambda_k(\theta)\tilde{x}^k$ and $\bar{\rho}^k = \theta^{-(\sigma_1^k+\delta_k)}\ \Omega_k(\theta)\ \tilde{\rho}^k$ for $k = 1,\ldots,q$, one obtains:

$$\begin{aligned}
\dot{\bar{x}}^k &= \Lambda_k(\theta)A_k\Lambda_k^{-1}(\theta)\bar{x}^k - \theta^{\delta_k}\Lambda_k(\theta)\Delta_k^{-1}(\theta)S_k^{-1}C_k^T C_k K(\tilde{x}^k) \\
&+ \Lambda_k(\theta)\left(g^k(u,\hat{x}) - g^k(u,x)\right) + \Lambda_k(\theta)\left(\Psi^k(u,\hat{x}) - \Psi^k(u,x)\right)\rho^k \\
&+ \theta^{-\sigma_1^k}\Delta_k(\theta)\Psi^k(u,\hat{x})\Omega_k^{-1}(\theta)\theta^{\delta_k+\sigma_1^k}\bar{\rho}^k - \theta^{\delta_k-\sigma_1^k}\Upsilon_{\hat{x}}^k P_k\ \Upsilon_{\hat{x}}^{k^T}C_k^T C_k K(\tilde{x}^k) \\
&= \theta^{\delta_k}A_k\bar{x}^k - \theta^{\delta_k-\sigma_1^k}S_k^{-1}C_k^T C_k K(\tilde{x}^k) + \Lambda_k(\theta)\left(g^k(u,\hat{x}) - g^k(u,x)\right) \\
&+ \Lambda_k(\theta)\left(\Psi^k(u,\hat{x}) - \Psi^k(u,x)\right)\rho^k + \theta^{\delta_k}\Delta_k(\theta)\Psi^k(u,\hat{x})\Omega_k^{-1}(\theta)\bar{\rho}^k \\
&- \theta^{\delta_k-\sigma_1^k}\Upsilon_{\hat{x}}^k P_k\ \Upsilon_{\hat{x}}^{k^T}C_k^T C_k K(\tilde{x}^k) \qquad (25)
\end{aligned}$$

$$\dot{\bar{\rho}}^k(t) = -\theta^{\delta_k}\theta^{-\sigma_1^k}P_k(t)\Upsilon_{\hat{x}}^{k^T}(t)C_k^T C_k k(\tilde{x}^k) \qquad (26)$$

substituting (26)in (25), one obtains:

$$\begin{aligned}
\dot{\bar{x}}^k &= \theta^{\delta_k}A_k\bar{x}^k - \theta^{\delta_k-\sigma_1^k}S_k^{-1}C_k^T C_k K(\tilde{x}^k) + \Upsilon_{\hat{x}}^k\ \dot{\bar{\rho}}^k + \Lambda_k(\theta)\left(g^k(u,\hat{x}) - g^k(u,x)\right) \\
&+ \Lambda_k(\theta)\left(\Psi^k(u,\hat{x}) - \Psi^k(u,x)\right)\rho^k + \theta^{\delta_k}\Delta_k(\theta)\Psi^k(u,\hat{x})\Omega_k^{-1}(\theta)\bar{\rho}^k \qquad (27)
\end{aligned}$$

Now, define: $\eta_k = \bar{x}^k - \Upsilon_{\hat{x}}^k\bar{\rho}^k$ where the matrix $\Upsilon_{\hat{x}}^k \in R^{n_k \times m_k}$ is governed by equation

(14) with $\xi \triangleq \hat{x}$. One can show that:

$$\begin{array}{rcl} \dot{\eta}_k & = & \dot{\bar{x}}^k - \dot{\Upsilon}^k_{\hat{x}}\overline{\rho}^k - \Upsilon^k_{\hat{x}}\dot{\overline{\rho}}^k \\ & = & \theta^{\delta_k}A_k(\eta_k + \Upsilon^k_{\hat{x}}\overline{\rho}^k) - \theta^{\delta_k-\sigma_1^k}S_k^{-1}C_k^T C_k K(\tilde{x}^k) + \Lambda_k(\theta)\left(g^k(u,\hat{x}) - g^k(u,x)\right) \\ & + & \Lambda_k(\theta)\left(\Psi^k(u,\hat{x}) - \Psi^k(u,x)\right)\rho^k + \theta^{\delta_k}\Delta_k(\theta)\Psi^k(u,\hat{x})\Omega_k^{-1}(\theta)\overline{\rho}^k - \dot{\Upsilon}^k_{\hat{x}}\overline{\rho}^k \\ & = & \theta^{\delta_k}A_k\eta_k + \theta^{\delta_k}S_k^{-1}C_k^T C_k\Upsilon^k_{\hat{x}}\overline{\rho}^k - \theta^{\delta_k-\sigma_1^k}S_k^{-1}C_k^T C_k K(\tilde{x}^k) \\ & + & \Lambda_k(\theta)\left(g^k(u,\hat{x}) - g^k(u,x)\right) + \Lambda_k(\theta)\left(\Psi^k(u,\hat{x}) - \Psi^k(u,x)\right)\rho^k \end{array} \quad (28)$$

Then, we consider the Lyapunov functions $V_1(\eta) = \overline{\eta}^T S \overline{\eta} = \sum_{k=1}^{q} V_{1k}(\eta_k)$ where $V_{1k}(\eta_k) = \eta^{k^T} S_k \eta^k$ and $S = diag(S_1, \ldots, S_q)$, $V_2(\overline{\rho}) = \overline{\rho}^T P^{-1}\overline{\rho} = \sum_{k=1}^{q} V_{2k}(\rho^k)$ and $V = V_1 + V_2$.

$$\begin{array}{rcl} \dot{V}_{1k} & = & 2\eta_k^T S_k \dot{\eta}_k \\ & = & 2\theta^{\delta_k}\eta_k^T S_{1k}A_k\eta_k + 2\theta^{\delta_k}\eta_k^T C_k^T C_k\Upsilon^k_{\hat{x}}\overline{\rho}^k - 2\theta^{\delta_k-\sigma_1^k}\eta_k^T C_k^T C_k K(\tilde{x}^k) \\ & + & 2\eta_k^T S_k\Lambda_k(\theta)\left(g^k(u,\hat{x}) - g^k(u,x)\right) + 2\eta_k^T S_k\Lambda_k(\theta)\left(\Psi^k(u,\hat{x}) - \Psi^k(u,x)\right)\rho^k \\ & = & -\theta^{\delta_k}V_{1k} + \theta^{\delta_k}\eta_k^T C_k^T C_k\eta_k + 2\theta^{\delta_k}\eta_k^T C_k^T C_k\Upsilon^k_{\hat{x}}\overline{\rho}^k - 2\theta^{\delta_k-\sigma_1^k}\eta_k^T C_k^T C_k K(\tilde{x}^k) \\ & + & 2\eta_k^T S_k\Lambda_k(\theta)\left(g^k(u,\hat{x}) - g^k(u,x)\right) + 2\eta_k^T S_k\Lambda_k(\theta)\left(\Psi^k(u,\hat{x}) - \Psi^k(u,x)\right)\rho^k \end{array} \quad (29)$$

According to(16), one obtains:

$$\begin{array}{rcl} \eta_k^T C_k^T C_k K(\tilde{x}^k) & = & \bar{x}^{k^T}\Delta_k^{-1}(\theta)C_k^T C_k K(\tilde{x}^k) - (\Upsilon^k_{\hat{x}}\overline{\rho}^k)^T C_k^T C_k K(\tilde{x}^k) \\ & = & (\Delta_k^{-1}(\theta)\bar{x}^k)^T C_k^T C_k K(\tilde{x}^k) - (\Upsilon^k_{\hat{x}}\overline{\rho}^k)^T C_k^T C_k K(\tilde{x}^k) \\ & = & \theta^{-\sigma_1^k}(\tilde{x}^k)^T C_k^T C_k K(\tilde{x}^k) - (\Upsilon^k_{\hat{x}}\overline{\rho}^k)^T C_k^T C_k K(\tilde{x}^k) \\ & \geq & \frac{1}{2}\theta^{-\sigma_1^k}\tilde{x}^{k^T}C_k^T C_k\tilde{x}^k - (\Upsilon^k_{\hat{x}}\overline{\rho}^k)^T C_k^T C_k K(\tilde{x}^k) \\ & \geq & \frac{1}{2}\theta^{-\sigma_1^k}\left(\Lambda_k^{-1}(\theta)\bar{x}^k\right)^T C_k^T C_k\Lambda_k^{-1}(\theta)\bar{x}^k - (\Upsilon^k_{\hat{x}}\overline{\rho}^k)^T C_k^T C_k K(\tilde{x}^k) \\ & \geq & \frac{1}{2}\theta^{\sigma_1^k}\bar{x}^{k^T}C_k^T C_k\bar{x}^k - (\Upsilon^k_{\hat{x}}\overline{\rho}^k)^T C_k^T C_k K(\tilde{x}^k) \\ & \geq & \frac{1}{2}\theta^{\sigma_1^k}\left(\eta_k + \Upsilon^k_{\hat{x}}\overline{\rho}^k\right)^T C_k^T C_k\left(\eta_k + \Upsilon^k_{\hat{x}}\overline{\rho}^k\right) - (\Upsilon^k_{\hat{x}}\overline{\rho}^k)^T C_k^T C_k K(\tilde{x}^k) \\ & \geq & \frac{1}{2}\theta^{\sigma_1^k}\left(\eta_k^T C_k^T C_k\eta_k + (\Upsilon^k_{\hat{x}}\overline{\rho}^k)^T C_k^T C_k\Upsilon^k_{\hat{x}}\overline{\rho}^k\right) \\ & + & \theta^{\sigma_1^k}\eta_k^T C_k^T C_k\Upsilon^k_{\hat{x}}\overline{\rho}^k - (\Upsilon^k_{\hat{x}}\overline{\rho}^k)^T C_k^T C_k K(\tilde{x}^k) \end{array} \quad (30)$$

Then,

$$
\begin{aligned}
\dot{V}_{1k} &\leq -\theta^{\delta_k}V_{1k}+\theta^{\delta_k}\eta_k^T C_k^T C_k\eta_k+2\theta^{\delta_k}\eta_k^T C_k^T C_k\Upsilon_{\hat{x}}^k\overline{\rho}^k-2\theta^{\delta_k}\eta_k^T C_k^T C_k\Upsilon_{\hat{x}}^k\overline{\rho}^k \\
&+ 2\eta_k^T S_{1k}\Lambda_k(\theta)\left(g^k(u,\hat{x})-g^k(u,x)\right)+2\eta_k^T S_{1k}\Lambda_k(\theta)\left(\Psi^k(u,\hat{x})-\Psi^k(u,x)\right)\rho^k \\
&- \theta^{\delta_k}\left(\eta_k^T C_k^T C_k\eta_k+(\Upsilon_{\hat{x}}^k\overline{\rho}^k)^T C_k^T C_k\Upsilon_{\hat{x}}^k\overline{\rho}^k\right)+2\theta^{\delta_k}\theta^{-\sigma_1^k}(\Upsilon_{\hat{x}}^k\overline{\rho}^k)^T C_k^T C_k K(\tilde{x}^k) \\
&\leq -\theta^{\delta_k}V_{1k}-\theta^{\delta_k}(\Upsilon_{\hat{x}}^k\overline{\rho}^k)^T C_k^T C_k\Upsilon_{\hat{x}}^k\overline{\rho}^k+2\theta^{\delta_k}\theta^{-\sigma_1^k}(\Upsilon_{\hat{x}}^k\overline{\rho}^k)^T C_k^T C_k K(\tilde{x}^k) \\
&+ 2\eta_k^T S_k\Lambda_k(\theta)\left(g^k(u,\hat{x})-g^k(u,x)\right)+2\eta_k^T S_k\Lambda_k(\theta)\left(\Psi^k(u,\hat{x})-\Psi^k(u,x)\right)\rho^k
\end{aligned}
$$

where:

• $\alpha_k = sup \left\{ \frac{\partial g_i^k}{\partial x_j^k}(u,x); x \in R^n and \|u\|_\infty \leq M \right\}$ and $\chi_{l,j}^{k,i} = 0$ if $\frac{\partial g_i^k}{\partial x_j^k}(u,x) \equiv 0$, $\chi_{l,j}^{k,i} = 1$ otherwise,

• $\beta_k = sup \left\{ \frac{\partial \Psi_i^k}{\partial x_j^k}(u,x); x \in R^n and \|u\|_\infty \leq M \right\}$ and $\varepsilon_{l,j}^{k,i} = 0$ if $\frac{\partial \Psi_i^k}{\partial x_j^k}(u,x) \equiv 0$, $\varepsilon_{l,j}^{k,i} = 1$ otherwise.

as, $|\vec{x}_j^l| \leq |\eta_j^l| + |\Upsilon_{\hat{x}_k}^{sl} \bar{\rho}_s^l|$, one obtains:

$$\begin{aligned} \dot{V}_k &\leq -\theta^{\delta_k} V_{1k} - \theta^{\delta_k} \theta^{\sigma_1^k} V_{2k} \\ &+ 2C_2 \sqrt{\lambda_{max}^k(S_k)} \sqrt{V_{1k}} \sum_{i=1}^{\lambda_k} \sum_{l=1}^{q} \sum_{j=1}^{\lambda_l} \sum_{s=1}^{m_k} \omega_{i,j}^{k,i} \theta^{\sigma_j^l - \sigma_i^k} \left(|\eta_j^l| + |\Upsilon_{\hat{x}_k}^{sl} \bar{\rho}^{sl}| \right) \end{aligned} \tag{36}$$

with, $\omega_{l,j}^{k,i} = \chi_{l,j}^{k,i} + \varepsilon_{l,j}^{k,i}$.
Now, we have,

$$\begin{aligned} \dot{V}_k &\leq -\theta^{\delta_k} V_{1k} - \theta^{\delta_k} \theta^{\sigma_1^k} V_{2k} \\ &+ 2C_2 \sqrt{\lambda_{max}^k(S_k)} \sqrt{\theta^{\delta_k} V_{1k}} \sum_{i=1}^{\lambda_k} \sum_{l=1}^{q} \sum_{j=1}^{\lambda_l} \omega_{i,j}^{k,i} \;\; \theta^{\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2}} \left(\frac{\sqrt{\theta^{\delta_l} V_{1l}}}{\sqrt{\lambda_{min}^l(S_k)}} + \frac{\sqrt{\theta^{\delta_l} V_{2l}}}{\sqrt{\lambda_{min}^l(P_l)}} \right) \end{aligned} \tag{37}$$

where $\lambda_{min}^l(S_l)$ is the minimum eigenvalue of S_l and $\lambda_{min}^l(P_l)$ is the minimum eigenvalue of P_l.
Now, according to the choice of the σ_1^k's given by (9), one can show that the following condition is satisfied [8]:

$$if \;\; \omega_{l,j}^{k,i} = 1 \;\; then \;\; \sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} \leq -\varepsilon < 0 \tag{38}$$

where $\varepsilon = \frac{\delta}{2^{q-1}} > 0$.
Now, assume that $\theta \geq 1$, then, one gets:

$$\begin{aligned}
\dot{V}_k &\leq -\theta^{\delta_k} V_{1k} - \theta^{\delta_k}\theta^{\sigma_1^k} V_{2k} \\
&+ 2C_2\sqrt{\lambda_{max}^k(S_k)}\sqrt{\theta^{\delta_k}V_{1k}} \sum_{i=1}^{\lambda_k}\sum_{l=1}^{q}\sum_{j=1}^{\lambda_l} \omega_{i,j}^{k,i}\ \theta^{-\varepsilon}\left(\frac{\sqrt{\theta^{\delta_l}V_{1l}}}{\sqrt{\lambda_{min}^l(S_k)}} + \frac{\sqrt{\theta^{\delta_l}V_{2l}}}{\sqrt{\lambda_{min}^l(P_l)}}\right) \\
&\leq -\theta^{\delta_k} V_{1k} - \theta^{\delta_k}\theta^{\sigma_1^k} V_{2k} \\
&+ 2n_kC_2\sqrt{\lambda_{max}^k(S_k)}\sqrt{\theta^{\delta_k}V_{1k}}\ \theta^{-\varepsilon} \sum_{l=1}^{q}\sum_{j=1}^{\lambda_l} \left(\frac{\sqrt{\theta^{\delta_l}V_{1l}}}{\sqrt{\lambda_{min}^l(S_k)}} + \frac{\sqrt{\theta^{\delta_l}V_{2l}}}{\sqrt{\lambda_{min}^l(P_l)}}\right) \\
&\leq -\theta^{\delta_k} V_{1k} - \theta^{\delta_k}\theta^{\sigma_1^k} V_{2k} + 2C_3\sqrt{\lambda_{max}^k(S_k)}\sqrt{\theta^{\delta_k}V_{1k}}\ \theta^{-\varepsilon}\sum_{l=1}^{q}\sum_{j=1}^{\lambda_l}\sqrt{\theta^{\delta_l}V_{1l}} \\
&+ 2C_4\sqrt{\lambda_{max}^k(S_k)}\sqrt{\theta^{\delta_k}V_{1k}}\ \theta^{-\varepsilon}\sum_{l=1}^{q}\sum_{j=1}^{\lambda_l}\sqrt{\theta^{\delta_l}V_{2l}} \\
&\leq -\theta^{\delta_k} V_{1k} - \theta^{\delta_k}\theta^{\sigma_1^k} V_{2k} + 2C_5\theta^{-\varepsilon}\sqrt{\theta^{\delta_k}V_{1k}}\ \sqrt{\theta^{\delta_k}V_1} \\
&+ 2C_6\theta^{-\varepsilon}\sqrt{\theta^{\delta_k}V_{1k}}\ \sqrt{\theta^{\delta_k}V_2} \\
&\leq -\theta^{\delta_k} V_{1k} - \theta^{\delta_k}\theta^{\sigma_1^k} V_{2k} + 2C_5\theta^{-\varepsilon}(\theta^{\delta_k}V_{1k})\ \sqrt{\theta^{\delta_k}V_1} \\
&+ 2C_6\theta^{-\varepsilon}\sqrt{\theta^{\delta_k}V_{1k}}\ \sqrt{\theta^{\delta_k}V_2}
\end{aligned} \tag{39}$$

Hence,

$$\dot{V} \leq -\left(\theta^{\delta_k} - 2C_5\theta^{-\varepsilon}\theta^{\delta_k}\right)V_1 - \theta^{\delta_k}\theta^{\sigma_1^k}V_2 + 2C_6\theta^{-\varepsilon}\sqrt{\theta^{\delta_k}V_1}\ \sqrt{\theta^{\delta_k}V_2} \tag{40}$$

Now, set $V_1^* = \theta^{\delta_k}(1-2C_5\theta^{-\varepsilon})V_1$, $V_2^* = \theta^{\delta_k}\theta^{\sigma_1^k}V_2$ and $V^* = V_1^* + V_2^*$. Notice that $\theta^{\delta}V \leq V^* \leq \theta^{\delta_1}V$, where δ, θ^{δ_1} are respectively given by (6). Then,

$$\begin{aligned}
\dot{V} &\leq -(V_1^* + V_2^*) + \frac{2C_6\theta^{-\varepsilon}}{\sqrt{1-2C_5 2C_6\theta^{-\varepsilon}}}(V_1^* + V_2^*) \\
&\leq -\left(1 - \frac{2C_6\theta^{-\varepsilon}}{\sqrt{1-2C_5\theta^{-\varepsilon}}}\right)(V_1^* + V_2^*)
\end{aligned} \tag{41}$$

Now, choosing θ_0 and δ such that $\left(1 - \frac{2C_6\theta^{-\varepsilon}}{\sqrt{1-2C_5\theta^{-\varepsilon}}}\right) > 0$ one obtains:

$$\dot{V} \leq -\left(1 - 2(C_5 + C_6)\theta^{-\varepsilon}\right)V \tag{42}$$

This completes the proof of theorem (2.1).

2.3 Some particular observers

Some particular expressions of the vector $K(\tilde{x}^{1k})$ that satisfy conditions (16) shall be given and discussed in this section.

2.3.1 Adaptive high gain observers

Consider the following expression of $K(\tilde{x}^k)$:

$$K_{HG}(\tilde{x}^k) \quad = \quad kK(\tilde{x}^k) \tag{43}$$

where $k \geq \frac{1}{2}$ is a real number. One can easily check that expression (43) satisfies conditions (16) . More specifically, the proposed observer with $K(\tilde{x}^k)$ specialized as in (43) is in fact an adaptive version of the well known high gain state observer.

2.3.2 Adaptive sliding mode like observers

At first glance, the following vector seems to be a potential candidate for the expression of $K(\tilde{x}^k)$:

$$K(\tilde{x}^k) \quad = \quad ksign(\tilde{x}^k) \tag{44}$$

where $k > 0$ is a real number and 'sign' is the usual sign function. Indeed, condition (16) is trivially satisfied by (44). However, expression (44) cannot be used due the discontinuity of sign function. Indeed, such discontinuity hampers the applicability of the Lyapunov approach used throughout the proof. In order to overcome this difficulty, one shall use continuous functions which have similar properties that those of the sign function. Of practical importance, these functions are widely used when implementing sliding mode observers. Indeed, consider the following function:

$$K_{Tanh}(\tilde{x}^k) \quad = \quad kTanh(\tilde{x}^k) \tag{45}$$

where *Tanh* denotes the hyperbolic tangent function and $k > 0$ is a real number.

It is easy to see that conditions (16) is satisfied for relatively high values of k.

Similarly to the hyperbolic tangent function, one can easily check that the inverse tangent function $K_{ArcTan}(\tilde{x}^k)$, the inverse sinus function $K_{Sinh}(\tilde{x}^k)$, etc., also constitute valid expressions for $K(\tilde{x}^k)$. Besides, one can consider new valid expressions for $K(\tilde{x}^k)$, for example by adding $K_{Tanh}(\tilde{x}^k)$ to $K_{HG}(\tilde{x}^k)$.

3 Conclusion

In this paper, we have discussed an adaptive observer for a class of MIMO uniformly observable nonlinear systems. A global exponential adaptive observer has then been proposed for these classes of systems. The gain of the parameter adaptation of the proposed observers involves a design function satisfying some mild conditions that have been given. Of fundamental importance, the global exponential convergence of these observers was shown to be guaranteed under the well known persistent excitation condition.

References

[1] G. Bastin and M. Gevers. Stable adaptive observers for nonlinear time varying systems. IEEE Trans. on Automat. Contr., 33:650-658, 1988.

[2] G. Besançon. Remarks on nonlinear adaptive observers design. Syst. Control Lett., 41(4):271-280, 2000.

[3] G. Bornard and H. Hammouri. A high gain observer for a class of uniformly observable systems. In Proc. 30th IEEE Conference on Decision and Control, Bringhton, England, 1991.

[4] Y. M. Cho and R. Rajamani. A systematic approach to adaptive observer synthesis for nonlinear systems. IEEE Trans. on Automat. Contr., 42:524-537, 1997.

[5] S. Drakunov. Sliding mode observers based on equivalent control method. In Proc. 31th IEEE Conference on Decision and Control, Tucson, Arizona, 1992.

[6] S.Drakunov and V. Utkin. Sliding mode observers. Tutorial. In Proc. 34th IEEE Conference on Decision and Control, New Orleans, LA, 1995.

[7] M. Farza, M. M'Saad, and L. Rossignol. Observer design for a class of MIMO nonlinear systems. Automatica, 40:135-143, 2004.

[8] F.L. Liu, M. Farza, M. M'Saad, and H. Hammouri. High Gain Observer based on coupled structures. Conference on Systems and Control (CSC'07). Marrakech, Morocco, mai 2007.

[9] A. Filipescu, L. Dugard, and J.M. Dion. Adaptive gain sliding controller for uncertain parameters nonlinear systems. application to flexible joint robots. In Proc. 42nd IEEE Conference on Decision and Control, Maui, HAwaii USA, 2003.

[10] J.P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems - application to bioreactors. IEEE Trans. on Aut. Control, 37:875-880, 1992.

[11] H. Hammouri and M. Farza. Nonlinear observers for locally uniformly observable systems. ESAIM: COCV, 9:353-370, 2003.

[12] Y. Koubaa, M.Farza et M. M'Saad : Synthèse d'observateurs adaptatifs pour une classe de systèmes non-linéaires, actes du CIFA, Douz, Tunisie, 2004.

[13] G. Lüders and K.S. Narendra. An adaptive observer and identifier for a linear system. IEEE Trans. on Automat. Contr., 18:496-499, 1973.

[14] T. Maatoug, M. Farza and M. M'Saad and Y. Koubaa. Adaptive Observer Design for a Class of MIMO Nonlinear Systems. In Proc. Third International Conference on Signals Systems Decision and Information Technology (SSD'07),Hammamet, Tunisie, 2007.

[15] R. Marino, G.L. Santosuosso, and P. Tomei. Robust Adaptive Observers for Nonlinears Systems with Bounded Disturbances. IEEE Trans. on Automat. Contr., 46:967-972, 2001.

[16] R. Marino and P. Tomei. Global adaptive observers for nonlinear systems via filtered transformations. IEEE Trans. on Automat. Contr., 37:1239-1245, 1992.

[17] R. Marino and P. Tomei. Adaptive observers with arbitrary exponential rate of convergence for nonlinear systems. IEEE Trans. on Automat. Contr., 40:1300-1304, 1995

[18] G. Kreisselmeier. Adaptive observers with exponential rate of convergence. IEEE trans. on Automatic Control, 22(1):2-8, 1977.

[19] R. Rajamani and J.K. Hedrick. Adaptive observer for active automotive suspensions - theory and experiment. IEEE Trans. on Control Systems Technology, 3:86-93, 1995.

[20] V.I. Utkin. Sliding mode in optimization and control. Springer-Verlag, 1992.

[21] A. Xu and Q. Zhang. State and parameter estimation for nonlinear systems. In Proc. of the 15th IFAC World Congress, 2002. Barcelona, Spain.

[22] Q. Zhang. Adaptive observers for MIMO linear time-varying systems. IEEE Trans. on Automat. Contr., 47:525-529, 2002.

[23] Q. Zhang and A. Clavel. Adaptive observer with exponential forgetting factor for linear time-varying systems. In Proc. 40th IEEE Conference on Decision and Control, 2001. Orlando, Florida.