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Design of parametric and state estimation algorithms for stochastic systems

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Abstract. *This paper is deals with the parametric and state estimation of the dynamic systems operating in a stochastic environment and represented by linear discrete-time state space mathematical models. We consider three situations of the state space mathematical model. In the first situation, we suppose that the parameters are unknown, but the state vector is known. A parametric estimation algorithm is developed. In the second situation, we suppose that the parameters are known, but the state vector is unknown. The formulation of the state estimation problem is treated by using the Kalman filter. In the third situation, we suppose that the parameters and the state vector are unknown. Thus, a parametric and state estimation algorithm is proposed. A numerical example is treated to validate the performances and limitations of the theoretical results.*

Key words: *Stochastic systems; State space mathematical models; Parametric estimation; State estimation; Recursive algorithms.*

1. Introduction

The formulation of the control problem (regulation, tracking, regulation and tracking conjointly) of a dynamic system consists on its description by a mathematical model. Two types of mathematical models are often used in the system control; the input-output mathematical models and the state space mathematical models. The content of the paper is limited to dynamic systems, which are described by state space mathematical models.

The development of an optimal control law for a dynamic system, which is described by a state space mathematical model, requires the knowledge of the state variables. However, in several practical situations, some or all these state variables cannot be determined by direct measurements. Notice that the dynamic system can be subjected to various types of random disturbances.

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Thus, and in order to develop an optimal control law, we must proceed, in a first step, to estimate the state variables of the system. This step of state estimation will be formulated from the knowledge of several measurements (input and output signals) on the system. It should be stressed that in the formulation of the state estimation problem, we suppose that the parameters intervening in the state space mathematical model are known. However, in several industrial applications, these parameters are partially or completely unknown, and consequently, a step of parametric estimation is necessary.

The object of this paper is the design of recursive algorithms allowing to estimate the parameters and the state variables of a dynamic systems operating in a stochastic environment and described by linear discrete-time state space mathematical models. The parameters and the state vector of state space mathematical model are supposed to be known or unknown. We must note that Kamoun (2007) examined the same problem of parametric and state estimation of stochastic systems.

The rest of this paper is organized as follows: In paragraph 2, we present the considered state space mathematical model and the posed parametric and/or state estimation problem of stochastic systems. Paragraph 3 is reserved for the parametric estimation of the considered state space mathematical model. Thus, a parametric estimation algorithm is developed. The stability analysis of this algorithm is treated using the Lyapunov method. The state estimation problem of the considered state space mathematical model is studied in paragraph 4. A recursive algorithm, which can estimate conjointly the parameters and the state variables of the considered state space mathematical model, is proposed in paragraph 5. A numerical example is treated in paragraph 6 and some conclusions are given in paragraph 7.

2. State space mathematical model

Let us consider a controlled stochastic system that can be represented by following state space mathematical model:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + v(k) \\ y(k) &= C^T x(k) + e(k) \end{aligned} \tag{1}$$

where $x(k) \in \Re^n$, $u(k)$ and $y(k)$ represent the state vector, the input and the output of the system at the discrete-time k, respectively, $v(k) \in \Re^n$ is the random disturbance vector which acts on the system, $e(k)$ indicates the random disturbance which affects the measurement of the output $y(k)$, A is the dynamic matrix of the system, and B and C are vectors, which are defined by:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \tag{2}$$

$$B^T = [b_1 \ldots b_n] \tag{3}$$

and

$$C^T = [1 \; 0 \ldots 0] \tag{4}$$

We assume that the system noise $v(k)$ is white Gaussien noise with a variance-covariance matrix Q and the measurement noise $e(k)$ is white Gaussien noise with a variance σ^2. Further, assume that the system noise $v(k)$ is not correlated with the measurement noise $e(k)$. Thus, we can write the following expressions:

$$\mathcal{E}[v(k)] = 0 \tag{5}$$

$$\mathcal{E}[v(k)v^T(j)] = Q\delta_{kj} \tag{6}$$

$$\mathcal{E}[e(k)] = 0 \tag{7}$$

and

$$\mathcal{E}[e^2(k)] = \sigma^2 \tag{8}$$

where $\mathcal{E}$ denotes the symbol of the expectation and δ_{kj} represents the symbol of Kronecker, i.e.: $\delta_{kj} = 0$, if $k \neq j$, and $\delta_{kj} = 1$, if $k = j$.

Notice that in practice, the variance-covariance matrix Q (system noise) and variance σ^2 (measurement noise) might change with each step of measurement. However here, we assume they are constant.

We can consider the three following situations of the state space mathematical model (1):

1. the parameters intervening in matrix A and the vector B are unknown. The variables of the state vector $x(k)$ are known at every discrete-time k. The difficulty that can arise, in this first situation, relates to estimate the parameters intervening in matrix A and the vector B;

2. the parameters intervening in matrix A and the vector B are known. However, the state vector $x(k)$ being unknown, i.e. the state variables are not available for measurement. The difficulty that can arise, in this second situation, consists to estimate this state vector;

3. the parameters intervening in matrix A and the vector B are unknown, and the state vector $x(k)$ being unknown. The difficulty that can arise, in this third situation, relates to estimate these parameters and this state vector.

The three situations of the state space mathematical model (1) will be studied in this paper, by developing recursive algorithms, which can estimate the parameters and the state variables of this mathematical model.

3. Parametric estimation

Let us consider, in this paragraph, the situation of the state space mathematical model (1) where the parameters intervening in matrix A and the vector B are supposed partially or completely unknown. However, the state vector $x(k)$ is supposed known, this at every discrete-time k.

This paragraph develops a parametric estimate algorithm, which can be used to estimate the parameters intervening in matrix A and the vector B of the following state equation of the mathematical model (1):

$$x(k+1) = Ax(k) + Bu(k) + v(k) \qquad (9)$$

We propose to formulate the posed parametric estimation problem of the state equation (9) by using the least squares method with adjustable model.

The state equation of the adjustable model is described by:

$$x_p(k+1) = \hat{A}(k+1)x(k) + \hat{B}(k+1)u(k) \qquad (10)$$

where $x_p(k) \in \Re^n$ is the state vector of the adjustable model, and $\hat{A}(k+1)$ and $\hat{B}(k+1)$ represent the estimate at the discrete-time $k+1$ of matrix A and the vector B, respectively.

We define the state error vector $\delta(k)$, which represents the difference between state vector $x(k)$ of the system and the state vector $x_p(k)$ of its adjustable model, by the following expression:

$$\delta(k) = x(k) - x_p(k) \qquad (11)$$

The formulation of the estimation problem of the parameters intervening in matrix A and the vector B of the state equation (9) can be made from the minimization of a quadratic criterion on the state error $\delta(k)$ and by using several measurements of the state variables $x_1(k), \ldots, x_n(k)$ and the input signal $u(k)$.

3.1. Parametric estimation algorithm

We propose the following parametric estimation algorithm, which can estimate the parameters intervening in matrix A and in the vector B of the state equation (9):

$$\begin{aligned}
\hat{A}(k+1) &= \hat{A}(k) + \zeta(k) G \delta(k) x^T(k-1) \\
\hat{B}(k+1) &= \hat{B}(k) + \zeta(k) G \delta(k) u(k-1) \\
\delta(k) &= x(k) - \hat{A}(k) x(k-1) - \hat{B}(k) u(k-1) \\
\zeta(k) &= \frac{l}{\lambda_G [u^2(k-1) + x^T(k-1) x(k-1)]}
\end{aligned} \tag{12}$$

where G is a symmetric positive definite matrix (i.e., $G = G^T > 0$), λ_G is the maximum eigenvalue of the matrix G and l is a positive parameter, which must be selected in order to ensure the stability of this parametric estimation algorithm (see Theorem 1, given hereafter).

The parameter l can be selected time-varying parameter (i.e., $l(k)$), this will be able to give more robustness (better capacity of parametric adaptation, etc.) to the parametric estimation algorithm (12). Notice that the beach of variation of the time-varying parameter $l(k)$ must be selected in an adequate way, in order to ensure the stability condition of the parametric estimation algorithm (12). Kamoun (2003) was proposed a procedure of recursive computation of the time-varying parameter $l(k)$.

It should be stressed that the parametric estimation algorithm (12) adapts better to the dynamic systems where the level of the noise $v(k)$ is small, i.e. the values of the components of the variance-covariance matrix Q are weak. This was proven on several numerical examples.

To appreciate globally the quality of the estimated parameters of the considered state equation (9) by using the parametric estimation algorithm (12), we can plot the evolution of the following parametric distance $d(k)$:

$$d(k) = \left[\sum_{r=1}^{n} \sum_{s=1}^{n} [\frac{a_{rs} - \hat{a}_{rs}(k)}{a_{rs}}]^2 + \sum_{r=1}^{n} [\frac{b_r - \hat{b}_r(k)}{b_r}]^2 \right]^{0.5} \tag{13}$$

Notice that this parametric distance $d(k)$ can be calculated only in an example of numerical simulation.

Theorem 1. Let us consider a dynamic system that can be described by the state equation (9), where the noise vector $v(k)$ is absent. We can estimate the unknown parameters intervening in matrix A and the vector B of this equation by using the parametric estimation algorithm (12). In order to ensure the stability of this algorithm, we must choose the parameter l , which respects the following condition:

$$0 < l < 2 \tag{14}$$

❑

Proof

The state error $\delta(k)$ intervening in the parametric estimation algorithm (12) can be described by the following expression:

$$\delta(k) = \tilde{A}(k)x(k-1) + \tilde{B}(k)u(k-1) \tag{15}$$

where $\tilde{A}(k) = A - \hat{A}(k)$ and $\tilde{B}(k) = B - \hat{B}(k)$ are parametric errors.

Let us consider a Lyapunov function $V(k)$ on the parametric errors, such as:

$$V(k) = \tilde{B}^T(k)\tilde{B}(k) + \mathrm{tr}[\tilde{A}^T(k)\tilde{A}(k)] \tag{16}$$

We can express the variation of this Lyapunov function, noted by: $\Delta V(k+1) = V(k+1) - V(k)$, as follows:

$$\begin{aligned}\Delta V(k+1) = {} & \tilde{B}^T(k+1)\tilde{B}(k+1) + \mathrm{tr}[\tilde{A}^T(k+1)\tilde{A}(k+1)] \\ & - \tilde{B}^T(k)\tilde{B}(k) - \mathrm{tr}[\tilde{A}^T(k)\tilde{A}(k)]\end{aligned} \tag{17}$$

The parametric errors $\tilde{A}(k+1)$ and $\tilde{B}(k+1)$ can be described by:

$$\tilde{A}(k+1) = \tilde{A}(k) - \zeta(k)G\delta(k)x^T(k-1) \tag{18}$$

and

$$\tilde{B}(k+1) = \tilde{B}(k) - \zeta(k)G\delta(k)u(k-1) \tag{19}$$

We can show easily that the variation $\Delta V(k+1)$ of the considered Lyapunov function can be expressed as follows:

$$\begin{aligned}\Delta V(k+1) = {} & -2\zeta(k)[u(k-1)\tilde{B}^T(k) + x^T(k-1)\tilde{A}(k)]G\delta(k) \\ & + \zeta^2(k)\delta^T(k)[u^2(k-1)G^TG + x^T(k-1)x(k-1)G^TG]\delta(k)\end{aligned} \tag{20}$$

or, in a compact form:

$$\Delta V(k+1) = -\delta^T(\mathrm{k})\Phi(\mathrm{k})\delta(\mathrm{k}) \tag{21}$$

where $\Phi(k)$ is a matrix, which is described by:

$$\Phi(k) = 2\zeta(k)G - \zeta^2(k)\rho^2(k-1)G^TG \tag{22}$$

where the parameter $\rho^2(k-1)$ is given by:

$$\rho^2(k-1) = u^2(k-1) + x^T(k-1)x(k-1) \tag{23}$$

Therefore, the stability condition of Lyapunov is given by:

$$\delta^T(k)[2\zeta(k)G - \zeta^2(k)\rho^2(k-1)G^TG]\delta(k) > 0 \tag{24}$$

From (24), it easy to obtain the following inequality:

$$\lambda_G \rho^2(k-1)\zeta(k) < 2 \tag{25}$$

The parametric gain $\zeta(k)$ is, by definition, positive definite. This makes it possible to confirm that the quantity $\lambda_G \rho^2(k-1)\zeta(k)$ is positive. Thus, we can express the inequality (25) as follows:

$$\zeta(k) = \frac{l}{\lambda_G \rho^2(k-1)} \tag{26}$$

where the parameter l must satisfy the condition (14). ❑

4. State estimation

In this paragraph, let us consider the situation of the considered state space mathematical model (1) where the parameters intervening in matrix A and the vector B are supposed perfectly known. However, the state vector $x(k)$ is supposed unknown, i.e. the state variables $x_1(k),\ldots,x_n(k)$ are not available for measurement. The estimation problem of these state variables will be studied, in what follows.

The state estimation problem of stochastic systems was attracted the attention of many researchers. Thus, several works have been developed and published in the literature by using the results of Kalman (1960), and Kalman and Bucy (1961). In a special number of the review *IEEE Transactions on Automatic Control* published in March 1983, are given several articles dealing with the application of the Kalman filter in various industrial fields (aeronautics and space, tracking of radars, navigation of ships, instrumentation, nuclear energy, etc). Among the many works on the subject, we can quote those of Zarchan and Musoff (2000), and Grewal and Andrews (2001). Let us add that Sorenson collected in its works, published in 1985 (Sorenson, 1985), a treating several articles of the theory and the application of the Kalman filter, who were published in various types of international reviews.

This paragraph is devoted to the state estimation of the mathematical model (1), while supposing known: the parameters of matrix A (2) and the vector B (3), the characteristics (average, variance-covariance matrix) of the noises $v(k)$ and $e(k)$, the initial state vector $x(0)$. The state estimation problem consists of the determination of a best estimate of the state vector $x(k)$ from the knowledge of several measurements of the output $y(k)$ and the input $u(k)$ of the considered system. We can define the information sequence $I(k)$ on the system as: $I(k) = \{u(k), y(k); k = 0,\ldots,M\}$, M being the number of measurements. Minimizing a bearing criterion on the estimation state error, which corresponds to the difference between the actual value of the state and that estimated, will carry the development of this best estimate. We seek, generally, to minimize the expected value of the square of the estimation state error. That is, on average, the state estimation algorithm, which is to be developed, gives the smallest possible estimation state error.

We define $\hat{x}(k/k-1) \in \Re^n$ to be the *a priori* state estimate at discrete-time k given knowledge of the information system to discrete-time $k-1$, and $\hat{x}(k/k) \in \Re^n$ to be the *a posteriori* state estimate at discrete-time k given the knowledge information system to the same discrete-time k. To simplify the notation, one will note by: $\hat{x}(k/k-1) = \hat{x}(k)$ and $\hat{x}(k/k) = \hat{x}^\circ(k)$.

We can define *a priori* $\xi(k)$ and *a posteriori* $\xi^\circ(k)$ estimate errors as, respectively:

$$\xi(k) = x(k) - \hat{x}(k) \tag{27}$$

and

$$\xi^\circ(k) = x(k) - \hat{x}^\circ(k) \tag{28}$$

The variance–covariance matrix $R(k)$ of the *a priori* estimate error is defined as follows:

$$R(k) = \mathcal{E}[\xi(k)\xi^T(k)] \tag{29}$$

The variance–covariance matrix $R^\circ(k)$ of the *a posteriori* estimate error is defined as follows:

$$R^\circ(k) = \mathcal{E}[\xi^\circ(k)\xi^{\circ T}(k)] \tag{30}$$

The development of the best estimate of the state vector $x(k)$, on the basis of the optimal statistical filtering of Kalman, consists of finding an equation that computes the *a posteriori* state estimate $\hat{x}^\circ(k)$ as a linear combination of *a priori* state estimate $\hat{x}(k)$ and a weighted difference between an actual measurement $y(k)$ and *a priori* prediction value $\hat{y}(k) = C^T\hat{x}(k)$, such that:

$$\hat{x}^\circ(k) = \hat{x}(k) + K(k)[y(k) - C^T\hat{x}(k)] \tag{31}$$

where $K(k)$ is a vector gain.

Looking at equation (31) we can note that the *a priori* prediction error $\varepsilon(k)$, as defined by: $\varepsilon(k) = y(k) - \hat{y}(k) = y(k) - C^T\hat{x}(k)$, which is also called innovation or residual, corresponds to the best estimate of the measurement noise $e(k)$.

Several forms for computing the vector gain $K(k)$ were proposed in the literature (see, e.g., Brown and Hwang, 1992), and this, according to the structure of the considered state space mathematical model. One form of the vector gain $K(k)$, which minimizes equation (30), is given by:

$$K(k) = R(k)C[C^T R(k)C + \sigma^2]^{-1} \quad (32)$$

which can be written as follows

$$K(k) = \frac{R(k)C}{\sigma^2 + C^T R(k)C} \quad (33)$$

Let us note that, if the variance σ^2 of the measurement noise $e(k)$ is very small (i.e., approaches zero), then the vector gain $K(k)$ weights the residual more heavily.

4.1. State estimation algorithm

The state estimation algorithm, which can estimate the state variables $x_1(k),\ldots,x_n(k)$ of the considered mathematical model (1), is constituted by the two following steps:

First step ″Measurement update (Correct, discrete-time k)″:

. compute the vector gain $K(k)$

$$K(k) = \frac{R(k)C}{\sigma^2 + C^T R(k)C} \quad (34)$$

. compute the state estimate vector $\hat{x}^{\circ}(k)$

$$\hat{x}^{\circ}(k) = \hat{x}(k) + K(k)[y(k) - C^T \hat{x}(k)] \quad (35)$$

. compute the variance-covariance matrix $R^{\circ}(k)$

$$R^{\circ}(k) = R(k) - K(k)C^T R(k) \quad (36)$$

Second step ″Time update (Predict, discrete-time $k+1$)″:

. compute the state estimate vector $\hat{x}(k+1)$

$$\hat{x}(k+1) = A\hat{x}^{\circ}(k) + Bu(k) \quad (37)$$

. compute the variance-covariance matrix $R(k+1)$

$$R(k+1) = AR^{\circ}(k)A^T + Q \quad (38)$$

The practical implementation of the first step requires the knowledge of two types of elements. They are the elements relating to the system (measurement of the output $y(k)$, vector C and variance σ^2) and the calculated elements (*a priori* state estimate vector $\hat{x}(k)$, variance–covariance matrix $R(k)$). Let us stress that these two elements were already calculated during the second step (in accordance with the second step), and this, with the iteration $k-1$. Let us add that during the first step, we compute the vector gain $K(k)$ and the variance–covariance matrix $R^{\circ}(k)$, which will be used in the second step. In fact, the equations (34)-(36) of the first step make it possible to correct the *a priori* state estimate vector $\hat{x}(k)$, and this, according to the difference between the measured value of the output $y(k)$ rebuilt starting from this state estimate. Thus, the first step is also called step of correction.

The practical implementation of the second step, which is also called step of prediction, requires the knowledge of two types of elements. They are the elements relating to the system (measurement of the input signal $u(k)$, vector B, matrix A and variance–covariance matrix Q) and the calculated elements in the first step (*a posteriori* state estimate vector $\hat{x}^{\circ}(k)$, variance–covariance matrix $R^{\circ}(k)$). Let us add that during this second step, we calculate the variance–covariance matrix $R(k+1)$. The *a priori* state estimate vector $\hat{x}(k+1)$ and the variance–covariance matrix $R(k+1)$ will be used in the first step, and this, at the iteration $k+1$. Let us note that the equations (37) and (38) of the second step allow to simulate the predicted value of the state from the estimated value $\hat{x}^{\circ}(k)$.

5. Parametric and state estimation

The last situation under consideration of the state space mathematical model (1) will be treated in this paragraph, where the parameters intervening in matrix A and the vector B are supposed unknown, and the state vector $x(k)$ is supposed unknown (i.e. the state variables $x_1(k),\ldots,x_n(k)$ are not available for measurement).

In fact, the configuration of the state space mathematical model, whose its parameters are unknown and its state variables are not available for measurement, corresponds in the majority of the cases which can arise in the industrial applications (see, e.g., Sorenson, 1985).

The object of this fifth paragraph is to develop a recursive algorithm, which can be estimated conjointly the unknown parameters intervening in matrix A and the vector B, and the not measurable variables of the state vector $x(k)$. Basing on the results developed in the two preceding paragraphs, we propose to formulate this parametric and state estimation problem.

5.1. Parametric and state estimation algorithm

We propose hereafter a recursive algorithm, which makes estimate conjointly the parameters and the state vector of the considered mathematical model (1). The proposed parametric and state estimation algorithm is constituted by the two following steps:

First step ″Estimation, Measurement update (Correct, discrete-time k)″:
. estimate the parameters intervening in matrix A and in the vector B

$$\begin{aligned}
&\hat{A}(k)=\hat{A}(k-1)+\zeta(k-1)G\delta(k-1)\hat{x}^{T}(k-2)\\
&\hat{B}(k)=\hat{B}(k-1)+\zeta(k-1)G\delta(k-1)u(k-2)\\
&\delta(k-1)=\hat{x}(k-1)-\hat{A}(k-1)\hat{x}(k-2)-\hat{B}(k-1)u(k-2)\\
&\zeta(k-1)=\frac{l}{\lambda_G[u^2(k-2)+\hat{x}^T(k-2)\hat{x}(k-2)]}
\end{aligned} \tag{39}$$

. compute the vector gain $K(k)$

$$K(k)=\frac{R(k)C}{\sigma^2+C^TR(k)C} \tag{40}$$

. compute the state estimate vector $\hat{x}^{\circ}(k)$

$$\hat{x}^{\circ}(k)=\hat{x}(k)+K(k)[y(k)-C^T\hat{x}(k)] \tag{41}$$

. compute the variance-covariance matrix $R^{\circ}(k)$

$$R^{\circ}(k)=R(k)-K(k)C^TR(k) \tag{42}$$

Second step ″Time update (Predict, discrete-time $k+1$)″:
. compute the state estimate vector $\hat{x}(k+1)$

$$\hat{x}(k+1)=\hat{A}(k)\hat{x}^{\circ}(k)+\hat{B}(k)u(k) \tag{43}$$

. compute the variance-covariance matrix $R(k+1)$

$$R(k+1)=\hat{A}(k)R^{\circ}(k)\hat{A}^T(k)+Q \tag{44}$$

The implementation of the proposed parametric and state estimation algorithm (39)-(44) requires the knowledge of the followings elements:

1. choice of the initial conditions for the parametric and state estimation algorithm (39)-(44): $\hat{A}(k-1)$, $\hat{B}(k-1)$, $\hat{x}(k)$, $\hat{x}(k-1)$, $\hat{x}(k-2)$, $\hat{x}^{\circ}(k)$, $R(k)$ and $R^{\circ}(k)$;

2. choices of the matrix gain G and the parameter l ;

3. knowledge of the variance σ^2 and the variance–covariance matrix Q ;

4. knowledge of several measurements (input $u(k)$ and output $y(k)$; $k=0,\ldots,M$) resulting from the considered system, and this, at each iteration k . It is supposed here that the number of measurements M is sufficiently large.

A diagram of the proposed parametric and state estimation algorithm (39)-(44) is represented in Figure 1.

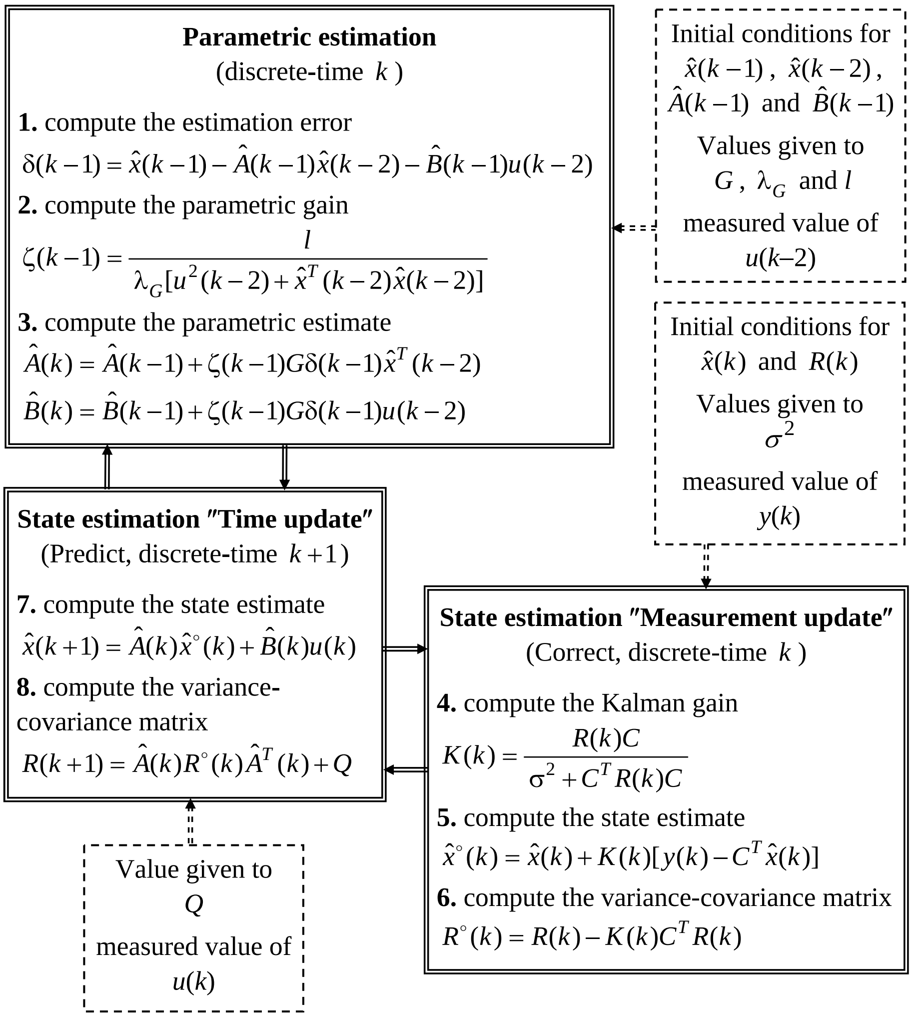

Figure 1. Diagram of the proposed parametric and state estimation algorithm (39)-(44).

6. Simulation results

We treat here a numerical example for testing the performances and limitations of the proposed parametric and state estimation algorithm (39)-(44).

Let us consider a dynamic system operating in a stochastic environment, which can be described by the following state space mathematical model:

$$x(k+1) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x(k) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(k) + v(k)$$
$$y(k) = [1\ \ 0]x(k) + e(k) \tag{45}$$

where $x^T(k) = [x_1(k)\ \ x_2(k)]$, $u(k)$ and $y(k)$ are the state vector, the input and the output of the system at the discrete-time k, respectively, $v^T(k) = [v_1(k)\ \ v_2(k)]$ is the random disturbance vector which acts on the system and $e(k)$ indicates the random disturbance which affects the measurement of the output $y(k)$.

We suppose that the six parameters a_{11}, a_{12}, a_{21}, a_{22}, b_1 and b_2 of the state space mathematical model (45) are unknown, and the two state variables $x_1(k)$ and $x_2(k)$ are not available for measurement. However, the signals of the input $u(k)$ and the output $y(k)$ are measurable. Thus, we can collect, at every discrete-time k, the values of $u(k)$ and $y(k)$ resulting from the considered system. The estimation of these six parameters and these two state variables will be make by using the parametric and state estimation algorithm (39)-(44).

The data concerning this numerical example are as follows:

1. the six parameters intervening in the state space mathematical model (45) are chosen, such as: $a_{11} = 0.38$, $a_{12} = 0.18$, $a_{21} = 0.28$, $a_{22} = -0.16$, $b_1 = 0.20$ and $b_2 = 0.34$;
2. the input signal $u(k)$ being a pseudo-random binary sequence of level $[+1,-1]$;
3. the random disturbances $v_1(k)$ and $v_2(k)$ are a white Gaussien noises. The variance-covariance matrix Q of the vector $v^T(k) = [v_1(k)\ \ v_2(k)]$ is given by: $Q = [0.0060\ 0\ ;\ 0\ 0.0030]$;
4. the random disturbance $e(k)$ is a white Gaussien noise with zero mean and variance $\sigma^2 = 0.1580$;
5. the number of measurements M being chosen, such as: $M = 160$ (i.e., $u(k), y(k); k = 0,\ldots,160$);
6. the initial conditions of the elements intervening in the parametric and state estimation algorithm (39)-(44) are chosen, such as: $l = 1.20$, $\hat{x}_1(0) = 0$, $\hat{x}_2(0) = 0$, $\hat{A}(0) = 0$, $\hat{B}(0) = 0$ and $G = [200\ 100\ ;\ 100\ 200]$..

The global quality of the six estimated parameters of the state space mathematical model (45) can be evaluated by considering the following parametric distance $d(k)$:

$$d(k) = \left[\sum_{r=1}^{2}\sum_{s=1}^{2}\left[\frac{a_{rs} - \hat{a}_{rs}(k)}{a_{rs}}\right]^2 + \sum_{r=1}^{2}\left[\frac{b_r - \hat{b}_r(k)}{b_r}\right]^2 \right]^{0.5} \tag{46}$$

Figure 2 gives the curves of the state errors $\delta_1(k)$ and $\delta_2(k)$, the parametric distance $d(k)$ and the trace of the variance-covariance matrix $R^{\circ}(k)$.

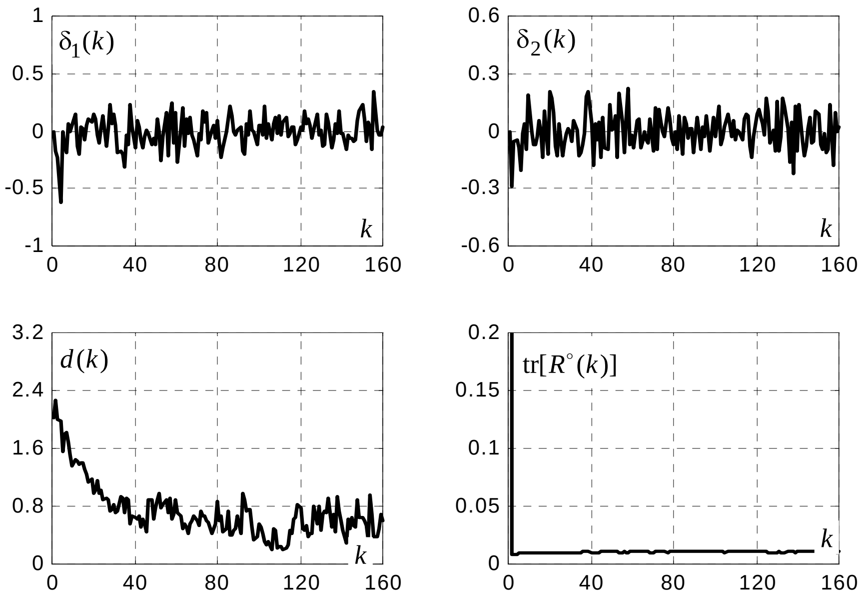

Figure 2. Curves of the state errors $\delta_1(k)$ and $\delta_2(k)$, the parametric distance $d(k)$ and the trace of the variance-covariance matrix $R^\circ(k)$.

The obtained results of this numerical example have showed the good performances, which is given by the proposed parametric and state estimation algorithm (39)-(44). In this context, and by examining more particularly the evolution of the various elements of Figure 2, we can affirm that qualities of parametric estimate and state estimate of the considered system considered are very good. Indeed, the parametric distance $d(k)$, which can be used like a criterion of global evaluation of the parametric estimate, decrease asymptotically (on statistical average) towards a minimal value. Let us add, that the state errors $\delta_1(k)$ and $\delta_2(k)$ are weak and that their variances are close to those of the random disturbances $v_1(k)$ and $v_2(k)$, respectively. We can notice that, from the curve of trace of the variance-covariance matrix $R^\circ(k)$ of the *a posteriori* estimation error, the quality of the state estimate is good. Indeed, the value of the trace of this variance-covariance matrix at discrete-time $k = 160$ is near of the value of the trace of the variance-covariance matrix Q ($\text{trace}[R^\circ(160)] = 0.0104 \approx \text{trace}[Q] = 0.0090$).

7. Conclusion

This paper was treated parametric and state estimation of dynamic systems operating in a stochastic environment. We have considered the stochastic systems that can be described by the class of linear discrete-time state space mathematical models. The parameters and the state vector of state space mathematical model are supposed to be known or unknown.

We have considered three situations of this class of state space mathematical models, which can arise in the industrial applications. Thus, the studied estimation problems, in this paper, are related to these three situations.

In the first situation, we have supposed that the parameters of the state space mathematical model are unknown and the state vector is known (i.e., the state variables are available for measurement). A parametric estimation algorithm was developed. The stability problem of this algorithm was treated by using the Lyapunov method. A necessary an sufficient condition for stabilising the developed parametric estimation algorithm was given.

In the second situation, we have supposed that the parameters of the state space mathematical model are known and the state vector is unknown (i.e., the state variables are not available for measurement). A state estimation algorithm is given by utilising the optimal filtering of Kalman, allowing the estimate of the state variables of state space mathematical model.

In the third situation, we have supposed that the parameters and the state vector of the state space mathematical model are unknown. A recursive algorithm, which allows the jointly estimation of these parameters and the state vector was proposed. Note that the practical implementation of this algorithm can be made easily. Notice that the proposed parametric and state estimation algorithm is constituted by two steps. The first step is relative to estimation and measurement update equations. However, the second step is relative to time update equations.

We have treated a numerical example in order to test the performances and limitations of the proposed parametric and state estimation algorithm. In this case, we have considered a dynamic system operating in a stochastic environment, which is described by a linear discrete-time state space mathematical model, where these parameters and state vector are supposed unknown. The obtained results of this numerical simulation are satisfactory and showed the good performances of estimate that can be obtained by the proposed parametric and state estimation algorithm.

An extension of these theoretical results to dynamic large-scale systems composed into interconnected systems is under work.

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